main

Volume 73B
Number 1
Jan.–Mar. 1969

P

APR 11 1969

Mathematical Sciences

Formerly—
Mathematics and
Mathematical Physics

MIAMI UNIVERSITY LIBRARY

U.S. DEPARTMENT OF COMMERCE

SECTION B

JOURNAL OF RESEARCH

NATIONAL BUREAU OF STANDARDS

Journal of Research

of the

National Bureau of Standards

B. Mathematical Sciences
(Formerly Mathematics and Mathematical Physics)

JAN.–MAR. • 1969

VOLUME • 73B

NUMBER • 1

Editor: Morris Newman

Associate Editors: John J. Sopka, F. W. Olver, and J. R. Edmonds

Publication dates: Mar. 10, June 10, Sept. 10, Dec. 10, 1969

JOURNAL OF RESEARCH

The National Bureau of Standards Journal of Research reports research and development in the fields of activity shown at right. Also included from time to time are survey articles on topics closely related to the Bureau's scientific and technical program.

The Journal is published in three separate sections as follows:

A. Physics and Chemistry

Contains papers of interest primarily to scientists working in these fields. Issued six times a year. Annual subscription: domestic, $5.00; foreign, $6.00. Single copy $1.00.

B. Mathematical Sciences.

Presents studies and compilations designed mainly for the mathematician and the theoretical physicist. Issued quarterly. Annual subscription: domestic, $2.25; foreign, $2.75. Single copy 75c.

C. Engineering and Instrumentation

Reports research and development results of interest chiefly to the engineer and the applied scientist. Issued quarterly. Annual subscription: domestic, $2.75; foreign, $3.50. Single copy 75c.

U.S. DEPARTMENT OF COMMERCE
Maurice H. Stans, Secretary
NATIONAL BUREAU OF STANDARDS
A. V. Astin, Director

Order all publications from the Superintendent of Documents
U.S. Government Printing Office, Washington, D.C. 20402.

Use of funds for printing this publication approved by the Director of the Bu
of the Budget (June 22, 1966).

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 73B, No. 1, January–March 1969

A Table of Integrals of the Error Functions*

Edward W. Ng and Murray Geller****

(October 23, 1968)

This is a compendium of indefinite and definite integrals of products of the Error function with elementary or transcendental functions. A substantial portion of the results are new.

Key Words: Astrophysics; atomic physics; Error functions; indefinite integrals; special functions; statistical analysis.

1. Introduction

Integrals of the error function occur in a great variety of applications, usually in problems involving multiple integration where the integrand contains exponentials of the squares of the arguments. Examples of applications can be cited from atomic physics [16],[1] astrophysics [13], and statistical analysis [15]. This paper is an attempt to give an up-to-date exhaustive tabulation of such integrals.

All formulas for indefinite integrals in sections 4.1, 4.2, 4.5, and 4.6 below were derived from integration by parts and checked by differentiation of the resulting expressions. Section 4.3 and the second half of 4.5 cover all formulas given in [7], with omission of trivial duplications and with a number of additions; section 4.4 covers essentially formulas given in [4], Vol. **I**, pp. 233–235. All these formulas have been re-derived and checked, either from the integral representation or from the hypergeometric series of the error function. Sections 4.7, 4.8 and 4.9 originated in a more varied way. Some formulas were derived from multiple integrals involving elementary functions, others from existing formulas for integrals of confluent hypergeometric functions, and still others, a small portion, were compiled directly from existing literature. In connection with the last three sections, the reader should refer to [3] and [4], Vol. **II**, pp. 402, 409–411.

Throughout this paper, we have adhered to the notations used in the NBS Handbook [9] and we have also assumed the reader's familiarity with the properties of the error functions, for which he is referred to [5]. In addition, the reader should also attend to the following conventions:

(i) $z=x+iy=r\exp(i\theta)$ is a complex variable,

$$\mathcal{R}(z)=x,\ \mathcal{I}(z)=y,\ |z|=r,\ \arg z=\theta;$$

(ii) the parameters a, b, and c are real and positive except where otherwise stated;

(iii) unless otherwise specified, the parameters n and k represent the integers 0, 1, 2 . . ., whereas the parameters p, q, and ν may be nonintegral;

(iv) the integration constants have been omitted for the indefinite integrals;

(v) when x is used (instead of z) as the integration variable, it means that the formula has been established only for real x, though it may still be valid for certain complex values;

(vi) the integration symbol $\int$ denotes a Cauchy principal value.

---

*An invited paper. This paper presents the results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract #NAS7-100, sponsored by the National Aeronautics and Space Administration.

**Present address: Jet Propulsion Laboratory, California Institute of Technology, Pasadena, Calif. 91103.

[1] Figures in brackets indicate the literature references at the end of this paper.

2. Glossary of Functions and Notation

$A(x)$	Gaussian Probability Function	$\frac{1}{\sqrt{2\pi}}\int_{-x}^{x} e^{-t^2/2}dt$
$C(z)$	Fresnel Integral	$\int_0^z \cos\left(\frac{\pi}{2}t^2\right)dt$
$Ci(z)$	Cosine Integral	$-\int_z^\infty \frac{\cos t}{t}dt$
$D_\nu(z)$	Parabolic Cylinder Function	
$e_n(z)$	Truncated Exponential	$\sum_{k=0}^{n}\frac{z^k}{k!}$
$-Ei(-z)\equiv E_1(z)$	Exponential Integral	$\int_z^\infty \frac{e^{-t}}{t}dt$
$Ei(x)$	Exponential Integral	$-\int_{-x}^{\infty} \frac{e^{-t}}{t}dt$
${}_1F_1(a;\ b;\ z)\equiv M(a,\ b,\ z)$	Confluent Hypergeometric Function	$\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!}$
${}_kF_l(a_1\ \ldots\ a_k;\ b_1\ \ldots\ b_l;\ z)$	Generalized Hypergeometric Function	$\sum_{n=0}^{\infty}\frac{(a_1)_n\ \ldots\ (a_k)_n}{(b_1)_n\ \ldots\ (b_l)_n}\frac{z^n}{n!}$
$H_n(x)$	Hermite Polynomial	
$\mathbf{H}_\nu(x)$	Struve Function	
$I_\nu(z)$	Modified Bessel Function	
$J_\nu(z)$	Bessel Function	
$K_\nu(z)$	Modified Bessel Function	
L_n^α	Generalized Laguerre Polynomial	
$M_{p,\,q}(z)$	Whittaker Function	
$Y_\nu(z)$	Neumann Function (Bessel Function of Second Kind)	
$P(x)$	Probability Function	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt$
$(p)_n$	Pochhammer's Symbol	$\Gamma(p+n)/\Gamma(p)$
$P_n(x)$	Legendre Polynomial	
$P_\nu^\mu(z)$	Associated Legendre Function of the First Kind	
$S(z)$	Fresnel Integral	$\int_0^z \sin\left(\frac{\pi}{2}t^2\right)dt$
$si(z)$	Sine Integral	$-\int_z^\infty \sin t\,\frac{dt}{t}$
$U(a,\ b,\ z)\equiv\Psi(a,\ b,\ z)$	Confluent Hypergeometric Function	
$W_{p,\,q}(z)$	Whittaker Function	

γ	Euler's Constant	0.5772156649 . . .
$\Gamma(p)$	Gamma Function	
$\gamma(p, z)$	Incomplete Gamma Function	$\int_0^z e^{-t}t^{p-1}dt$
$\Gamma(p, z)$	Incomplete Gamma Function	$\int_z^\infty e^{-t}t^{p-1}dt$
$\zeta(z)$	Riemann's Zeta Function	$\sum_{k=0}^{\infty} k^{-z}$
$\psi(z)$	Psi Function	$\frac{d}{dz}[\ln \Gamma(z)]$

3. Definition and Integral Representations

3.1. Definitions and Other Notations

1. $\operatorname{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}dt,$

2. $\operatorname{erfc}(z) \equiv \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}dt = 1 - \operatorname{erf}(z),$

3. $\operatorname{erfi}(z) \equiv -i \operatorname{erf}(iz) = \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2}dt.$

Some authors use the above notations without the factor $\frac{2}{\sqrt{\pi}}$, and some use $\Phi(z)$ for $\operatorname{erf}(z)$.

4. $w(z) \equiv e^{-z^2} \operatorname{erfc}(-iz).$

For real x, Dawson's integral is defined as

5. $F(x) \equiv \frac{\sqrt{\pi}}{2} e^{-x^2} \operatorname{erfi}(x) = \frac{\sqrt{\pi}}{2} \mathcal{I}[w(x)].$

The error function is also closely related to the Gaussian probability functions:

6. $\operatorname{erf}(x) = 2P(x\sqrt{2}) - 1 = A(x\sqrt{2}).$

3.2. Integral Representations

1. $\operatorname{erf}(az) = \frac{2az}{\sqrt{\pi}} \int_0^1 e^{-a^2z^2t^2}\, dt$

2. $\operatorname{erfc}(az) = \frac{2}{\sqrt{\pi}} e^{-a^2z^2} \int_0^\infty e^{-(t^2+2azt)}dt$

3. $\operatorname{erf}\left(az + \frac{w}{a}\right) = \frac{2a}{\sqrt{\pi}} \exp\left(c - \frac{w^2}{a^2}\right) \int e^{-(a^2z^2+2wz+c)}dz.$

4. $\operatorname{erfc}\left(\frac{z}{a}\right)=\frac{2a}{\sqrt{\pi}}\exp\left(c-\frac{z^2}{a^2}\right)\int_0^\infty e^{-(a^2t^2+2zt+c)}dt$

5. $e^{2ab}\operatorname{erf}\left(ax+\frac{b}{x}\right)+e^{-2ab}\operatorname{erf}\left(ax-\frac{b}{x}\right)=\frac{4a}{\sqrt{\pi}}\int e^{-a^2x^2-b^2/x^2}dx$

6. $\operatorname{erfc}(az)=\frac{2a}{\sqrt{\pi}}\,e^{-a^2z^2}\int_0^\infty \frac{e^{-a^2t^2}t\,dt}{(z^2+t^2)^{1/2}}, \qquad \mathcal{R}(a)>0,\ \mathcal{R}(z)>0.$

7. $\operatorname{erfc}(az)=\frac{2z}{\pi}\,e^{-a^2z^2}\int_0^\infty \frac{e^{-a^2t^2}dt}{(t^2+z^2)}, \qquad \mathcal{R}(a)>0,\ |\arg z|<\pi,\ z\neq 0.$

8. $1-[\operatorname{erf}(x)]^2=\frac{4}{\pi}e^{-x^2}\int_0^1 \frac{e^{-x^2t^2}dt}{(t^2+1)}, \qquad x>0$

9. $\operatorname{erf}\left(x+\frac{iy}{2}\right)+\operatorname{erf}\left(x-\frac{iy}{2}\right)=\frac{4}{\sqrt{\pi}}\,e^{y^2/4}\int e^{-x^2}\cos xy\,dx$

10. $\operatorname{erf}\left(x+\frac{iy}{2}\right)-\operatorname{erf}\left(x-\frac{iy}{2}\right)=\frac{4}{i\sqrt{\pi}}\,e^{y^2/4}\int e^{-x^2}\sin xy\,dx$

11. $\operatorname{erf}(x)=\frac{x}{\sqrt{\pi}}\int_0^\pi e^{x^2\cos\theta}\cos(x^2\sin\theta+\theta/2)d\theta, \qquad x\neq 0$

12. $\operatorname{erf}(x)=\frac{1}{\pi}\int_0^\infty e^{-t}\sin(2x\sqrt{t})\,\frac{dt}{t}.$

4. Integrals of Product of Error Functions With Other Functions

4.1. Combination of Error Function With Powers

1. $\int \operatorname{erf}(az)dz=z\operatorname{erf}(az)+\frac{1}{a\sqrt{\pi}}\exp(-a^2z^2)$

2. $\int \operatorname{erfc}(az)dz=z\operatorname{erfc}(az)-\frac{1}{a\sqrt{\pi}}\exp(-a^2z^2)$

3. $\int_0^\infty \operatorname{erfc}(ax)dx=\frac{1}{a\sqrt{\pi}}, \qquad |\arg a|<\frac{\pi}{4}$

4. $\int \operatorname{erf}(az)z\,dz=\frac{1}{2}z^2\operatorname{erf}(az)-\frac{1}{4a^2}\operatorname{erf}(az)+\frac{z}{2a\sqrt{\pi}}\exp(-a^2z^2)$

5. $\int \operatorname{erfc}(az)z\,dz=\frac{1}{2}z^2\operatorname{erfc}(az)+\frac{1}{4a^2}\operatorname{erf}(az)-\frac{z}{2a\sqrt{\pi}}\exp(-a^2z^2)$

6. $\int_0^\infty \operatorname{erfc}(ax)x\,dx=\frac{1}{4a^2}, \qquad |\arg a|<\frac{\pi}{4}$

7. $$\int \operatorname{erf}(az) z^n dz = \frac{z^{n+1}}{n+1} \operatorname{erf}(az) + \frac{e^{-a^2z^2}}{a\sqrt{\pi}(n+1)} \sum_{k=0}^{l-1} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}-k+1\right)} \frac{z^{n-2k}}{a^{2k}}$$

$$- \frac{1-j}{n+1} \frac{\Gamma\left(l+\frac{1}{2}\right)}{a^{n+1}\sqrt{\pi}} \operatorname{erf}(az), \qquad j=0 \text{ or } 1,\ 2l-j=n+1$$

8. $$\int \operatorname{erf}(az) z^{n+2} dz = \frac{(n+2)(n+1)}{2(n+3)a^2} \int \operatorname{erf}(az) z^n dz + \left(z^2 - \frac{n+2}{2a^2}\right) \frac{z^{n+1}}{(n+3)} \operatorname{erf}(az)$$

$$+ \frac{1}{a\sqrt{\pi}(n+3)} z^{n+2} e^{-a^2z^2}$$

9. $$\int \operatorname{erfc}(az) z^n dz = \frac{z^{n+1}}{(n+1)} \operatorname{erfc}(az) - \frac{e^{-a^2z^2}}{a\sqrt{\pi}(n+1)} \sum_{k=0}^{l-1} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}-k+1\right)} \frac{z^{n-2k}}{a^{2k}}$$

$$+ \frac{1-j}{n+1} \frac{\Gamma\left(l+\frac{1}{2}\right)}{a^{n+1}\sqrt{\pi}} \operatorname{erf}(az), \qquad j=0 \text{ or } 1,\ 2l-j=n+1$$

10. $$\int \operatorname{erfc}(az) z^{n+2} dz = \frac{(n+2)(n+1)}{2(n+3)a^2} \int \operatorname{erfc}(az) z^n dz + \left(z^2 - \frac{n+2}{2a^2}\right) \frac{z^{n+1}}{(n+3)} \operatorname{erfc}(az)$$

$$- \frac{1}{a\sqrt{\pi}(n+3)} z^{n+2} e^{-a^2z^2}$$

11. $$\int_0^\infty \operatorname{erfc}(ax) x^n dx = \frac{\Gamma\left(\frac{n}{2}+1\right)}{(n+1)\sqrt{\pi}a^{n+1}}, \qquad |\arg a| < \frac{\pi}{4}$$

12.[2] $$\int \operatorname{erf}(az) z^{-1} dz = \ln z \operatorname{erf}(az) - \frac{2a}{\sqrt{\pi}} \int \ln z e^{-a^2z^2} dz$$

13. $$\int \operatorname{erfc}(az) z^{-1} dz = \ln z \operatorname{erfc}(az) + \frac{2a}{\sqrt{\pi}} \int \ln z e^{-a^2z^2} dz$$

14. $$\int \operatorname{erf}(az) z^{-n} dz = -\frac{\operatorname{erf}(az)}{(n-1)z^{n-1}} + \frac{2a}{(n-1)\sqrt{\pi}} \int \frac{1}{z^{n-1}} e^{-a^2z^2} dz, \qquad n \geq 2$$

15. $$\int \operatorname{erfc}(az) z^{-n} dz = -\frac{\operatorname{erfc}(az)}{(n-1)z^{n-1}} - \frac{2a}{(n-1)\sqrt{\pi}} \int \frac{1}{z^{n-1}} e^{-a^2z^2} dz, \qquad n \geq 2$$

16. $$\int \operatorname{erf}(az) z^p dz = \frac{z^{p+1}}{p+1} \operatorname{erf}(az) - \frac{1}{(p+1)a^{p+1}\sqrt{\pi}} \gamma\left(\frac{p}{2}+1, a^2z^2\right), \qquad p > -2,\ p \neq -1$$

[2] See appendix for integrals on the right-hand sides of eqs (12 to 15).

17. $\int \operatorname{erfc}(az) z^p dz = \frac{z^{p+1}}{p+1} \operatorname{erfc}(az) + \frac{1}{(p+1)a^{p+1}\sqrt{\pi}} \gamma\left(\frac{p}{2}+1, a^2z^2\right), \qquad p > -1$

18. $\int_0^\infty \operatorname{erfc}(ax) x^p dx = \frac{1}{(p+1)a^{p+1}\sqrt{\pi}} \Gamma\left(\frac{p}{2}+1\right), \qquad |\arg a| < \frac{\pi}{4}, \; p > -1$

19. $\int_0^\infty \operatorname{erf}(ax) x^{p-2} dx = \frac{a^{1-p}}{\sqrt{\pi}(1-p)} \Gamma\left(\frac{p}{2}\right), \qquad |\arg a| < \frac{\pi}{4}, \; 0 < p < 1.$

4.2. Combination of Error Functions With Exponentials and Powers [3]

1. $\int \operatorname{erf}(az) e^{bz} dz = \frac{1}{b} e^{bz} \operatorname{erf}(az) - \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \operatorname{erf}\left(az - \frac{b}{2a}\right)$

2. $\int \operatorname{erfc}(az) e^{bz} dz = \frac{1}{b} e^{bz} \operatorname{erfc}(az) + \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \operatorname{erf}\left(az - \frac{b}{2a}\right)$

3. $\int_0^\infty \operatorname{erf}(ax) e^{-bx} dx = \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \operatorname{erfc}\left(\frac{b}{2a}\right), \qquad \Re(b) > 0, \; |\arg a| < \pi/4$

4. $\int_0^\infty \operatorname{erfc}(ax) e^{bx} dx = \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \left[1 + \operatorname{erf}\left(\frac{b}{2a}\right)\right] - \frac{1}{b}, \; |\arg(b-a)| < \frac{\pi}{4}$

5. $\int \operatorname{erf}(az) e^{bz} z dz = \frac{1}{b} \operatorname{erf}(az) e^{bz}\left(z - \frac{1}{b}\right) - \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \left\{ \left(\frac{b}{2a^2} - \frac{1}{b}\right) \operatorname{erf}(t) - \frac{1}{a\sqrt{\pi}} e^{-t^2} \right\}, \; t = az - \frac{b}{2a}$

6. $\int \operatorname{erfc}(az) e^{bz} z dz = \frac{1}{b} \operatorname{erfc}(az) e^{bz}\left(z - \frac{1}{b}\right)$

$$+ \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \left\{ \left(\frac{b}{2a^2} - \frac{1}{b}\right) \operatorname{erf}(t) - \frac{1}{a\sqrt{\pi}} e^{-t^2} \right\}, \; t = az - \frac{b}{2a}$$

7. $\int_0^\infty \operatorname{erf}(ax) e^{-bx} x dx = \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \left[\frac{1}{b} - \frac{b}{2a^2}\right] \operatorname{erfc}\left(\frac{b}{2a}\right) + \frac{1}{ab\sqrt{\pi}}, \; \Re(b) > 0, \; |\arg a| < \frac{\pi}{4}$

8. $\int_0^\infty \operatorname{erfc}(ax) e^{bx} x dx = \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \left[\frac{b}{2a^2} - \frac{1}{b}\right] \left[1 + \operatorname{erf}\left(\frac{b}{2a}\right)\right] + \frac{1}{b^2} + \frac{1}{ab\sqrt{\pi}}, \; |\arg(b-a)| < \frac{\pi}{4}$

9.[4] $\int \operatorname{erf}(az) e^{bz} z^n dz = (-1)^n \frac{n!}{b^{n+1}} e^{bz} \operatorname{erf}(az) e_n(-bz)$

$$- \frac{2a}{\sqrt{\pi}} (-1)^n \frac{n!}{b^{n+1}} \sum_{k=0}^{n} \frac{(-b)^k}{k!} \int z^k \exp(-a^2z^2 + bz) dz$$

[3] In this section a and b can take any value on the complex plane other than the origin, except where otherwise stated.

[4] See appendix for the integrals on the right-hand sides of eqs (9 to 12).

10. $$b\int \operatorname{erf}(az)e^{bz}z^n dz + n\int \operatorname{erf}(az)e^{bz}z^{n-1}dz$$
$$= e^{bz}z^{n-1}\left[z\operatorname{erf}(az) + \frac{1}{a\sqrt{\pi}}e^{-a^2z^2}\right]$$
$$-\frac{1}{a\sqrt{\pi}}\int (bz^n + nz^{n-1})\exp(-a^2z^2 + bz)dz$$

11. $$\int \operatorname{erfc}(az)e^{bz}z^n dz = (-1)^n\frac{n!}{b^{n+1}}e^{bz}\operatorname{erfc}(az)e_n(-bz)$$
$$+\frac{2a}{\sqrt{\pi}}\frac{(-1)^n n!}{b^{n+1}}\sum_{k=0}^{n}\frac{(-b)^k}{k!}\int z^k\exp(-a^2z^2 + bz)dz$$

12. $$b\int \operatorname{erfc}(az)e^{bz}z^n dz + n\int \operatorname{erfc}(az)e^{bz}z^{n-1}dz$$
$$= e^{bz}z^{n-1}\left[z\operatorname{erfc}(az) - \frac{1}{a\sqrt{\pi}}e^{-a^2z^2}\right]$$
$$+\int (bz^n + nz^{n-1})\exp(-a^2z^2 + bz)dz$$

13. $$\int_0^\infty \operatorname{erf}(ax)e^{-bx}x^n dx = \left(\frac{2}{\pi}\right)^{1/2}\sum_{k=0}^{n}\frac{n!}{b^{k+1}}(2a)^{\frac{k-n}{2}}\exp\left(\frac{b^2}{8a^2}\right)D_{-n+k-1}\left(\frac{b}{a\sqrt{2}}\right)$$
$$= \sum_{k=0}^{n}\frac{(-1)^{n-k}}{b^{k+1}}\frac{n!(a)^{k-n}}{(n-k)!}\frac{d^{n-k}}{dq^{n-k}}\left[\exp\left(\frac{q^2}{4}\right)\operatorname{erfc}\left(\frac{q}{2}\right)\right],$$
$$q = \frac{b}{a}, \qquad \mathscr{R}(b) > 0, \qquad |\arg a| < \frac{\pi}{4}$$

14. $$\int_0^\infty \operatorname{erfc}(ax)e^{bx}x^n dx = (-1)^{n+1}\frac{n!}{b^{n+1}}$$
$$+\left(\frac{2}{\pi}\right)^{1/2}\sum_{k=0}^{n}(-1)^k\frac{n!}{b^{k+1}}(2a^2)^{\frac{k-n}{2}}\exp\left(\frac{b^2}{8a^2}\right)D_{-n+k-1}\left(-\frac{b}{a\sqrt{2}}\right)$$
$$|\arg(b-a)| < \frac{\pi}{4}.$$

4.3. Combination of Error Function With Exponentials of More Complicated Arguments

1. $$\int_0^a e^{-x^2}\operatorname{erf}(x)dx = \frac{\sqrt{\pi}}{4}(\operatorname{erf} a)^2$$

2. $$\int_0^\infty \operatorname{erf}(ax)e^{-b^2x^2}dx = \frac{\sqrt{\pi}}{2b} - \frac{1}{b\sqrt{\pi}}\tan^{-1}\frac{b}{a}$$

3. $$\int_0^\infty \operatorname{erfc}(ax)e^{b^2x^2}dx = \frac{1}{2\sqrt{\pi}\,b}\ln\left[\frac{a+b}{a-b}\right], \qquad b \text{ may be complex}, \qquad |\arg a| < \frac{\pi}{4}$$

4. $\int_0^\infty \operatorname{erf}(ax)e^{-b^2x^2}x\,dx=\frac{a}{2b^2}(a^2+b^2)^{-1/2}, \qquad \Re(b^2)>\Re(a^2), \qquad \Re(b^2)>0$

5. $\int_0^\infty \operatorname{erfc}(ax)e^{b^2x^2}x\,dx=\frac{1}{2b^2}\left[\frac{a}{(a^2-b^2)^{1/2}}-1\right], \qquad \Re(a^2)>\Re(b^2)$

6. $\int_0^\infty \operatorname{erf}(ax)e^{-b^2x^2}x^2\,dx=\frac{\sqrt{\pi}}{4b^3}-\frac{1}{2\sqrt{\pi}}\left[\frac{1}{b^3}\tan^{-1}\frac{b}{a}-\frac{a}{b^2(a^2+b^2)}\right], \qquad |\arg a|<\frac{\pi}{4}$

7. $\int_0^\infty \operatorname{erfc}(ax)e^{-b^2x^2}x^2\,dx=\frac{1}{2\sqrt{\pi}}\left[\frac{1}{b^3}\tan^{-1}\frac{b}{a}-\frac{a}{b^2(a^2+b^2)}\right], \qquad |\arg a|<\frac{\pi}{4}$

8. $\int_0^\infty \operatorname{erf}(ax)e^{-b^2x^2}x^p\,dx=\frac{a}{\sqrt{\pi}}b^{-p-2}\Gamma\left(\frac{p}{2}+1\right){}_2F_1\left(\frac{1}{2},\frac{p}{2}+1;\frac{3}{2};-\frac{a^2}{b^2}\right),$

$\Re(b^2)>0, \qquad \Re(p)>-2$

9. $\int_0^\infty \operatorname{erfc}(ax)e^{b^2x^2}x^p\,dx=\frac{\Gamma(\frac{1}{2}p+1)}{\sqrt{\pi}\,(p+1)a^{p+1}}{}_2F_1\left(\frac{p+1}{2},\frac{p+2}{2};\frac{p+3}{2};\frac{b^2}{a^2}\right),$

$\Re(b^2)<\Re(a^2), \qquad \Re(p)>-1$

10. $\int_0^\infty \operatorname{erfc}(x)e^{x^2}x^{p-1}\,dx=\frac{1}{2}\sec\left(\frac{p\pi}{2}\right)\Gamma\left(\frac{p}{2}\right), \qquad 0<p<1$

11. $\int_0^\infty \operatorname{erf}(ax)e^{-b^2x^2}\frac{dx}{x}=\frac{1}{2}\ln\frac{(a^2+b^2)^{1/2}+a}{(a^2+b^2)^{1/2}-a}$

$$=\ln\frac{a+(a^2+b^2)^{1/2}}{b}, \qquad \Re(b^2)>0$$

12. $\int_0^\infty \operatorname{erf}(iax)e^{-a^2x^2-bx}\,dx=\frac{1}{2ai\sqrt{\pi}}\exp\left(\frac{b^2}{4a^2}\right)Ei\left(-\frac{b^2}{4a^2}\right), \qquad \Re(b)>0, \qquad |\arg a|<\frac{\pi}{4}$

13. $\int_{-\infty}^\infty \operatorname{erf}(x)e^{-(ax+b)^2}\,dx=-\frac{\sqrt{\pi}}{a}\operatorname{erf}\left(\frac{b}{\sqrt{a^2+1}}\right), \qquad \Re(a^2)>0$

14. $\int_0^\infty \operatorname{erf}(ix)e^{-(x^2+ax)}x\,dx=\frac{i}{\sqrt{\pi}}\left[\frac{1}{a}+\frac{a}{4}Ei\left(-\frac{a^2}{4}\right)\exp\left(\frac{a^2}{4}\right)\right], \qquad \Re(a)>0$

15. $\int_0^\infty \operatorname{erf}(ax)e^{-bx^4}x^3\,dx=\frac{a^2}{8\sqrt{\pi}\,b^{3/2}}\exp\left(\frac{a^4}{8b}\right)K_{1/4}\left(\frac{a^4}{8b}\right)$

16. $\int_1^\infty \operatorname{erfc}(ax)e^{a^2x}x^{-3}\,dx=\frac{1}{2}(1-2a^2)e^{a^2}\operatorname{erfc}(a)+\frac{a}{\sqrt{\pi}}$

17. $\int_1^\infty \operatorname{erfc}(ax)e^{a^2x}\frac{(x-1)^{p/2-1}}{x^{p+1}}\,dx=\frac{1}{\sqrt{\pi}}\,e^{a^2/2}2^{5p/4-2}a^{p/2-1}\Gamma\left(\frac{p}{2}\right)D_{-1-p}(a\sqrt{2}), \qquad p>0$

18. $\int_0^1 \operatorname{erfc}\left(\frac{ax}{\sqrt{2}}\right)e^{\frac{a^2x^2}{2}}x^{p-1}(1-x^2)^{-(p+1)/2}\,dx=\frac{1}{\sqrt{\pi}}\Gamma(p)\Gamma\left(\frac{1-p}{2}\right)2^{-p/2}e^{a^2/4}D_{-p}(a), \qquad 0<p<1$

19. $\int_0^a \operatorname{erf}(x) e^{x^2}(a^2-x^2)^{-1/2} x\,dx = \frac{\sqrt{\pi}}{2}(e^{a^2}-1), \qquad a>0$

20. $\int_0^a \operatorname{erfc}(x) e^{x^2}(a^2-x^2)^{-1/2} x\,dx = \frac{\sqrt{\pi}}{2}[1-e^{a^2}\operatorname{erfc}(a)], \qquad a>0$

21. $\int_0^a \operatorname{erf}(x) e^{x^2}(a^2-x^2)^{p-1} x\,dx = \frac{1}{2} e^{a^2} \frac{\Gamma(p)}{\Gamma(p+\frac{1}{2})} \gamma(p+\tfrac{1}{2}, a^2), \qquad p>0$

22. $\int_0^\infty \operatorname{erf}\left(\frac{a}{x}\right) e^{-b^2x^2} x\,dx = \frac{1}{2b^2}(1-e^{-2ab}), \qquad \mathscr{R}(a)>0, \qquad \mathscr{R}(b^2)>0$

23. $\int_0^\infty \operatorname{erfc}\left(\frac{a}{x}\right) e^{-b^2x^2} x\,dx = \frac{1}{2b^2} e^{-2ab}, \qquad \mathscr{R}(a)>0, \qquad \mathscr{R}(b^2)>0$

24. $\int_0^\infty \operatorname{erfc}\left(\frac{a}{x}\right) e^{-b^2x^2} \frac{dx}{x} = -Ei(-2ab), \qquad \mathscr{R}(a)>0, \qquad \mathscr{R}(b^2)>0$

25. $\int_0^\infty \operatorname{erfc}(ax) e^{-\frac{b^2}{4x^2}} x\,dx = \frac{1}{4a^2} e^{-ab}(1+ab) - \frac{b^2}{4}[-Ei(-ab)]$

26. $\int_0^\infty \operatorname{erf}(ax)[1-e^{-\frac{b^2}{4x^2}}] \frac{dx}{x} = \gamma + \ln(ab) + [-Ei(-ab)]$

27. $\int_0^\infty \operatorname{erfc}(ax) e^{-\frac{b^2}{x^2}} \frac{dx}{x} = -Ei(-2ab), \qquad \mathscr{R}(a)>0, \qquad \mathscr{R}(b^2)>0$

28. $\int_0^\infty \operatorname{erf}(ax) e^{-\frac{b^2}{4x^2}} \frac{dx}{x^3} = \frac{2}{b^2}(1-e^{-ab})$

29. $\int_0^\infty \operatorname{erfc}(ax) e^{-\frac{b^2}{4x^2}} \frac{dx}{x^3} = \frac{2}{b^2} e^{-ab}$

30. $\int_0^\infty \operatorname{erfc}\left(\frac{1}{x}\right) \exp\left(\frac{1}{x^2}-b^2x^2\right) dx = \frac{1}{b\sqrt{\pi}}[\sin 2bCi(2b) - \cos 2bsi(2b)]$

31. $\int_0^\infty \operatorname{erfc}\left(\frac{1}{x}\right) \exp\left(\frac{1}{x^2}-b^2x^2\right) x\,dx = \frac{\pi}{2b}[\mathbf{H}_1(2b) - Y_1(2b)] - \frac{1}{b}, \qquad |\arg b| < \frac{\pi}{4}$

32. $\int_0^\infty \operatorname{erfc}\left(\frac{1}{x}\right) \exp\left(\frac{1}{x^2}-b^2x^2\right) \frac{dx}{x} = \frac{\pi}{2}[\mathbf{H}_0(2b) - Y_0(2b)], \ |\arg b| < \frac{\pi}{4}$

33. $\int_0^\infty \left[\operatorname{erfc}\left(\frac{a}{x}\right)(x^2+2a^2) - \frac{2}{\sqrt{\pi}} axe^{-a^2/x^2}\right] e^{-b^2x^2} x\,dx = \frac{1}{2b^4} e^{-2ab}, \qquad |\arg b| < \frac{\pi}{4}, \ \mathscr{R}(a)>0$

34. $\int_0^\infty \operatorname{erfc}\left(ax+\frac{b}{x}\right) e^{-c^2x^2} x\,dx$

$$= \tfrac{1}{2}(a^2+c^2)^{-1/2}[a+(a^2+c^2)^{1/2}]^{-1} \exp[-2b(a+\sqrt{a^2+c^2})], \qquad \mathscr{R}(b)>0, \ \mathscr{R}(a^2+c^2)>0$$

35. $\int_0^\infty \left\{2\cosh ab - e^{-ab}\operatorname{erf}\left(\frac{b-2ax^2}{2x}\right) - e^{ab}\operatorname{erf}\left(\frac{b+2ax^2}{2x}\right)\right\} e^{-(c^2-a^2)x^2} x\,dx = \frac{1}{c^2-a^2} e^{-bc},$

$$a>0, \ b>0, \ \mathscr{R}(c^2)>0$$

36. $\int_0^\infty \cosh(2bx)\exp[(a\cosh x)^2]\operatorname{erfc}(a\cosh x)dx$

$$= \tfrac{1}{2}\sec(b\pi)e^{a^2/2}K_b(a^2), \qquad \mathscr{R}(a) > 0, -\tfrac{1}{2} < \mathscr{R}(b) < \tfrac{1}{2}$$

37. $\int_0^\infty \{\exp[-(x-a)^2] - \exp[-(x+a)^2]\}\operatorname{erf}(x)dx = \sqrt{\pi}\operatorname{erf}(a/\sqrt{2})$

38. $\int_0^\infty \{\exp[-(x-a)^2] + \exp[-(x+a)^2]\}\operatorname{erf}(x)\,dx = \dfrac{\sqrt{\pi}}{2}\{1 + [\operatorname{erf}(a/\sqrt{2})]^2\}.$

4.4 Definite Integrals From Laplace Transforms Involving Erf $(\sqrt{ax})$

1. $\int_0^\infty e^{-bx}[e^{ax}\operatorname{erf}(\sqrt{ax})]dx = (b-a)^{-1}(a/b)^{1/2}, \qquad b > a$

2. $\int_0^\infty e^{-bx}[e^{ax}\operatorname{erfc}(\sqrt{ax})dx] = b^{-1/2}(\sqrt{b}+\sqrt{a})^{-1}$

3. $\int_0^\infty e^{-bx}[e^{ax}\operatorname{erf}(\sqrt{ax}) - 2(ax/\pi)^{1/2}]dx = (b-a)^{-1}(a/b)^{3/2}, \qquad b > a$

4. $\int_0^\infty e^{-bx}[(\pi x)^{-1/2} - a^{1/2}e^{ax}\operatorname{erfc}(\sqrt{ax})]dx = (\sqrt{b}+\sqrt{a})^{-1}$

5. $\int_0^\infty e^{-bx}[1 - e^{ax}\operatorname{erfc}(\sqrt{ax})]dx = \sqrt{a}b^{-1}(\sqrt{b}+\sqrt{a})^{-1}$

6. $\int_0^\infty e^{-bx}[e^{ax}\operatorname{erfc}(\sqrt{ax}) + 2(ax/\pi)^{1/2} - 1]dx = a(\sqrt{b}+\sqrt{a})^{-1}b^{-3/2}$

7. $\int_0^\infty e^{-bx}[1 - 2(ax/\pi)^{1/2} + (2ax-1)e^{ax}\operatorname{erfc}(\sqrt{ax})]dx = ab^{-1}(\sqrt{b}+\sqrt{a})^{-2}$

8. $\int_0^\infty e^{-bx}[(x/\pi)^{1/2} - a^{1/2}xe^{ax}\operatorname{erfc}(\sqrt{ax})]dx = (4b)^{-1/2}(\sqrt{b}+\sqrt{a})^{-2}$

9. $\int_0^\infty e^{-bx}[8axe^{ax}\operatorname{erfc}(\sqrt{ax}) - 8(ax/\pi)^{1/2} + 1]dx = b^{-1}\left(\dfrac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}}\right)^2$

10. $\int_0^\infty e^{-bx}[e^{ax}\sqrt{ax}\ (2ax+3)\operatorname{erfc}(\sqrt{ax}) - 2(x/\pi)^{1/2}(ax+1)] = -(\sqrt{b}+\sqrt{a})^{-3}$

11. $\int_0^\infty e^{-bx}[(2a^2x^2+5ax+1)e^{ax}\operatorname{erfc}(\sqrt{ax}) - 2(ax+2)(ax/\pi)^{1/2}] = \sqrt{b}(\sqrt{b}+\sqrt{a})^{-3}$

12. $\int_0^\infty e^{-bx}[2(8a^2x^2+8ax+1)e^{ax}\operatorname{erf}(\sqrt{ax}) - 8(ax/\pi)^{1/2}(2ax+1) - 1]dx$

$$= b^{-1}(\sqrt{b}-\sqrt{a})^3(\sqrt{b}+\sqrt{a})^{-3}, \; b > a$$

13. $\int_0^\infty e^{-bx}[(2a^{1/2}x^{1/2}+1)xe^{ax}\operatorname{erf}(\sqrt{ax}) - 2(ax^3/\pi)^{1/2}]dx = b^{-1/2}(\sqrt{b}+\sqrt{a})^{-3}, \qquad b > a$

14. $\int_0^\infty e^{-bx}[(4a^2x^2+12ax+3)xe^{ax}\ \text{erfc}\ (\sqrt{ax})-2(a^3x^5/\pi)^{1/2}(2ax+5)]dx=3(\sqrt{b}+\sqrt{a})^{-4}$

15. $\int_0^\infty e^{-bx}[ae^{ax}\ \text{erfc}\ (\sqrt{ax})+\sqrt{ac}e^{cx}\ \text{erfc}\ (\sqrt{cx})-ce^{cx}]dx=(a-c)(b-c)^{-1}\sqrt{b}(\sqrt{b}+\sqrt{a})^{-1}$

16. $\int_0^\infty e^{-bx}[a^{1/2}e^{cx}\ \text{erf}\ (\sqrt{bx})+c^{1/2}e^{ax}\ \text{erfc}\ (\sqrt{ax})-c^{1/2}e^{cx}]dx$

$$=(a-c)\sqrt{c}(b-c)^{-1}b^{-1/2}(\sqrt{b}+\sqrt{a})^{-1},\ b>c$$

17. $\int_0^\infty e^{-bx}\left[2\left(\frac{x}{\pi}\right)^{1/2}e^{-a^2/(4x)}-a\ \text{erfc}\left(\frac{a}{2\sqrt{x}}\right)\right]dx=\frac{e^{-a\sqrt{b}}}{b^{3/2}}.$

18. $\int_0^\infty e^{-bx}\left[a\left(\frac{x}{\pi}\right)^{1/2}e^{-a^2/(4x)}+\left(x+\frac{a^2}{2}\right)\text{erf}\left(\frac{a}{2\sqrt{x}}\right)-\frac{a^2}{2}\right]dx=\frac{1}{b^2}(1-e^{-a\sqrt{b}}).$

4.5. Combination of Error Function With Trigonometric Functions

1. $\int \text{erf}\ (az)\sin bzdz=-\frac{1}{b}\cos bz\ \text{erf}\ (az)+\frac{1}{2b}\exp\left(-\frac{b^2}{4a^2}\right)\left\{\text{erf}\left(az-i\frac{b}{2a}\right)+\text{erf}\left(az+\frac{ib}{2a}\right)\right\}$

2. $\int \text{erf}\ (az)\cos bzdz=\frac{1}{b}\sin bz\ \text{erf}\ (az)+\frac{i}{2b}\exp\left(-\frac{b^2}{4a^2}\right)\left\{\text{erf}\left(az-\frac{ib}{2a}\right)-\text{erf}\left(az+\frac{ib}{2a}\right)\right\}$

3. $\int \text{erfc}\ (az)\sin bzdz=-\frac{1}{b}\cos bz\ \text{erfc}\ (az)-\frac{1}{2b}\exp\left(-\frac{b^2}{4a^2}\right)\left\{\text{erf}\left(az-\frac{ib}{2a}\right)+\text{erf}\left(az+\frac{ib}{2a}\right)\right\}$

4. $\int \text{erfc}\ (az)\cos bzdz=\frac{1}{b}\sin bz\ \text{erfc}\ (az)-\frac{i}{2b}\exp\left(-\frac{b^2}{4a^2}\right)\left\{\text{erf}\left(az-\frac{ib}{2a}\right)-\text{erf}\left(az+\frac{ib}{2a}\right)\right\}$

5. $\int_0^\infty \text{erfc}\ (ax)\sin bxdx=\frac{1}{b}\left[1-\exp\left(-\frac{b^2}{4a^2}\right)\right],\qquad |\arg a|<\frac{\pi}{4}$

6. $\int_0^\infty \text{erfc}\ (ax)\cos bxdx=-\frac{i}{b}\exp\left(-\frac{b^2}{4a^2}\right)\text{erf}\left(\frac{ib}{2a}\right),\qquad |\arg a|<\frac{\pi}{4}$

7. $\int_0^\infty \text{erfc}\ (ax)\cos (bx)xdx=\frac{1}{2a^2}\exp\left(-\frac{b^2}{4a^2}\right)-\frac{1}{b^2}\left[1-\exp\left(-\frac{b^2}{4a^2}\right)\right]$

8. $\int_0^\infty \text{erfc}\ (\sqrt{ax})\sin bxdx=\frac{1}{b}-\left(\frac{a/2}{a^2+b^2}\right)^{1/2}[(a^2+b^2)^{1/2}-a]^{-1/2},\qquad \mathscr{R}(a)>|\mathscr{I}(b)|$

9. $\int_0^\infty \text{erfc}\ (\sqrt{ax})\cos bxdx=\left(\frac{a/2}{a^2+b^2}\right)^{1/2}[(a^2+b^2)^{1/2}+a]^{-1/2},\qquad \mathscr{R}(a)>|\mathscr{I}(b)|$

10. $\int_0^\infty \text{erf}\ (ax)\sin b^2x^2dx=\frac{1}{4b\sqrt{2\pi}}\left(\ln\frac{a^2+b^2+ab\sqrt{2}}{a^2+b^2-ab\sqrt{2}}+2\tan^{-1}\frac{ab\sqrt{2}}{b^2-a^2}\right),\qquad a>0$

11. $\int_0^\infty \operatorname{erfc}(ax)\sin bx x^p dx = \frac{\Gamma\left(\frac{p+3}{2}\right) b}{a^{p+2}\sqrt{\pi}(p+2)}\, {}_2F_2\left(\frac{p+2}{2}, \frac{p+3}{2}; \frac{3}{2}, \frac{p+4}{2}; -\frac{b^2}{4a^2}\right),$

$\Re(a) > 0, \Re(p) > 0$

12. $\int_0^\infty \operatorname{erfc}(ax)\cos bx x^p dx = \frac{\Gamma\left(\frac{p}{2}+1\right)}{a^{p+1}\sqrt{\pi}(p+1)}\, {}_2F_2\left(\frac{p+1}{2}, \frac{p+2}{2}; \frac{1}{2}, \frac{p+3}{2}; -\frac{b^2}{4a^2}\right),$

$\Re(a) > 0, \Re(p) > 0$

13. $\int_0^\infty \operatorname{erf}(ax)\frac{\sin bx}{x^2} dx = \frac{b}{2}\left[-Ei\left(-\frac{b^2}{4a^2}\right)\right] + \sqrt{\pi}\operatorname{erf}\left(\frac{b^2}{4a^2}\right)$

14. $\int_0^\infty \operatorname{erf}(ax)\frac{\cos bx}{x} dx = \frac{1}{2}\left[-Ei\left(-\frac{b^2}{4a^2}\right)\right]$

15. $\int_0^\infty [\operatorname{erfc}(ax) - \operatorname{erfc}(bx)]\frac{\cos px}{x} dx = \frac{1}{2}\left\{Ei\left(-\frac{p^2}{4a^2}\right) - Ei\left(-\frac{p^2}{4b^2}\right)\right\}$

16. $\int_0^\infty \operatorname{erfc}\left(\sqrt{\frac{a}{x}}\right)\sin bx dx = \frac{1}{b}\exp\left[-(2ab)^{1/2}\right]\cos\left[(2ab)^{1/2}\right], \quad \Re(a) > 0, \Re(b) > 0$

17. $\int_0^\infty \operatorname{erf}\left(\sqrt{\frac{a}{x}}\right)\cos bx dx = -\frac{1}{b}\exp\left[-(2ab)^{1/2}\right]\sin\left[(2ab)^{1/2}\right], \quad \Re(a) > 0, \Re(b) > 0$

18. $\int_0^\infty \operatorname{erfc}(ax)\tan x dx = \sum_{k=1}^{\infty}\frac{(-1)^k}{k}\exp\left(-\frac{k^2}{a^2}\right) + \ln 2$

19. $\int_0^\infty \operatorname{erfc}(ax)\sin^2 bx \frac{dx}{x} = \frac{1}{4}\left\{\gamma + 2\ln\left(\frac{b}{a}\right) + \left[-Ei\left(-\frac{b^2}{a^2}\right)\right]\right\}$

20. $\int_0^\infty \operatorname{erf}(ax)\sin bx \sin cx \frac{dx}{x} = \frac{1}{4}\left\{\left[-Ei\left(-\frac{(c-b)^2}{4a^2}\right)\right] - \left[-Ei\left(-\frac{(c+b)^2}{4a^2}\right)\right]\right\}, \quad c \neq b$

21. $\int_0^\infty \operatorname{erfc}(ax)\sin bx \sin cx \frac{dx}{x} = \frac{1}{2}\ln\left|\frac{c+b}{c-b}\right| - \int_0^\infty \operatorname{erf}(ax)\sin bx \sin cx \frac{dx}{x}, \quad c \neq b$

22. $\int_0^\infty \operatorname{erf}(ax)\cos bx \cos cx \frac{dx}{x} = \frac{1}{4}\left\{\left[-Ei\left(-\frac{(c-b)^2}{4a^2}\right)\right] + \left[-Ei\left(-\frac{(c+b)^2}{4a^2}\right)\right]\right\}, \quad c \neq b$

23. $\int_0^\infty \operatorname{erfc}(ax)\sin bx e^{a^2x^2} dx = \frac{\sqrt{\pi}}{2a}\exp\left(\frac{b^2}{4a^2}\right)\operatorname{erfc}\left(\frac{b}{2a}\right)$

24. $\int_0^\infty \operatorname{erf}(iax)\sin bx e^{-a^2x^2} dx = \frac{i\sqrt{\pi}}{2a}\exp\left(-\frac{b^2}{4a^2}\right)$

25. $\int_0^\infty \operatorname{erfc}(ax)\cos bx e^{a^2x^2} dx = \frac{1}{2a\sqrt{\pi}}\exp\left(\frac{b^2}{4a^2}\right)\left[-Ei\left(-\frac{b^2}{4a^2}\right)\right]$

26. $\int_0^\infty \operatorname{erfc}(ax)\sin x \cosh x dx = \frac{1}{2}\left[\sin\left(\frac{1}{2a^2}\right) - \cos\left(\frac{1}{2a^2}\right) + 1\right]$

27. $\int_0^\infty \operatorname{erfc}(ax)\cos x \sinh x\,dx = \frac{1}{2}\left[\sin\left(\frac{1}{2a^2}\right)+\cos\left(\frac{1}{2a^2}\right)-1\right]$

28. $\int_0^\infty \operatorname{erfc}(ax)\, x\cos x\cosh x\,dx = \left[\frac{1}{2a^2}\cos\left(\frac{1}{2a^2}\right)-\frac{1}{2}\sin\left(\frac{1}{2a^2}\right)\right]$

29. $\int_0^\infty \operatorname{erfc}(ax)\, x\sin x\sinh x\,dx = \left[\frac{3}{2}\cos\left(\frac{1}{2a^2}\right)+\frac{1}{a^2}\sin\left(\frac{1}{2a^2}\right)-\frac{3}{2}\right]$

30. $\int_0^\infty \left[e^{-bx}\operatorname{erfc}\left(ab-\frac{x}{2a}\right)-e^{bx}\operatorname{erfc}\left(ab+\frac{x}{2a}\right)\right]\sin px\,dx = \frac{2p}{(p^2+b^2)}\exp\left[-a^2(b^2+p^2)\right].$

4.6. Combination of Error Function With Logarithms and Powers

1. $\int \operatorname{erf}(az)\ln z\,dz = (\ln z-1)\left[z\operatorname{erf}(az)+\frac{1}{a\sqrt{\pi}}e^{-a^2z^2}\right]-\frac{1}{2a\sqrt{\pi}}Ei(-a^2z^2)$

2. $\int \operatorname{erfc}(az)\ln z\,dz = (\ln z-1)\left[z\operatorname{erfc}(az)-\frac{1}{a\sqrt{\pi}}e^{-a^2z^2}\right]+\frac{1}{2a\sqrt{\pi}}Ei(-a^2z^2)$

3. $\int_0^\infty \operatorname{erfc}(ax)\ln x\,dx = -\frac{1}{a\sqrt{\pi}}\left[1+\frac{\gamma}{2}+\ln a\right]$

4.[5] $\int \operatorname{erf}(az)z\ln z\,dz = \frac{1}{2}z^2\operatorname{erf}(az)\ln z+\frac{1}{2a\sqrt{\pi}}z\ln z e^{-a^2z^2}$
$-\frac{1}{2}\int z\operatorname{erf}(az)\,dz-\frac{1}{4a^2}\operatorname{erf}(az)-\frac{1}{2a\sqrt{\pi}}\int \ln z e^{-a^2z^2}dz$

5. $\int \operatorname{erfc}(az)z\ln z\,dz = \frac{1}{2}z^2\operatorname{erf}(az)\ln z-\frac{z\ln z}{2a\sqrt{\pi}}e^{-a^2z^2}$
$-\frac{1}{2}\int z\operatorname{erfc}(az)\,dz+\frac{1}{4a^2}\operatorname{erf}(az)+\frac{1}{2a\sqrt{\pi}}\int \ln z e^{-a^2z^2}dz$

6. $\int_0^\infty \operatorname{erfc}(ax)x\ln x\,dx = \frac{1}{8a^2}+\frac{1}{2a\sqrt{\pi}}\int_0^\infty \ln x e^{-a^2x^2}dx$

7. $(k+1)\int \operatorname{erf}(az)z^k\ln z\,dz = z^{k+1}\operatorname{erf}(az)\ln z+\frac{1}{a\sqrt{\pi}}z^k\ln z e^{-a^2z^2}$
$-\int z^k\operatorname{erf}(az)\,dz-\frac{1}{a\sqrt{\pi}}\int z^{k-1}e^{-a^2z^2}dz$
$-\frac{k}{a\sqrt{\pi}}\int z^{k-1}\ln z e^{-a^2z^2}dz$

8. $(k+1)\int \operatorname{erfc}(az)z^k\ln z\,dz = z^{k+1}\operatorname{erfc}(az)\ln z-\frac{1}{a\sqrt{\pi}}z^k\ln z e^{-a^2z^2}$
$-\int z^k\operatorname{erfc}(az)\,dz+\frac{1}{a\sqrt{\pi}}\int z^{k-1}e^{-a^2z^2}dz$
$+\frac{k}{a\sqrt{\pi}}\int z^{k-1}e^{-a^2z^2}\ln z\,dz$

[5] For the elementary integrals in eqs. (4 to 8), see appendix.

9. $(k+1)\int_0^\infty \operatorname{erfc}(ax)x^k \ln x\,dx = \frac{\Gamma(k/2)}{2\sqrt{\pi}a^{k+1}}\left[\frac{1}{k+1}+\frac{k}{2}\psi\left(\frac{k}{2}\right)-k\ln a\right]$.

4.7. Combination of Two Error Functions

1. $\int_0^\infty \operatorname{erf}(bx)\operatorname{erfc}(ax)dx = \frac{1}{b\sqrt{\pi}}\left(\frac{\sqrt{a^2+b^2}}{a}-1\right)$

2. $\int_0^\infty \operatorname{erfc}(bx)\operatorname{erfc}(ax)dx = \frac{1}{ab\sqrt{\pi}}(a+b-\sqrt{a^2+b^2})$

3. $\int_0^\infty \operatorname{erf}\left(\frac{b}{x}\right)\operatorname{erfc}(ax)dx$

$$= \int_0^\infty \operatorname{erf}(bx)\operatorname{erfc}\left(\frac{a}{x}\right)\frac{dx}{x^2} = \frac{1}{a\sqrt{\pi}}(1-e^{-2ab}) - \frac{2b}{\sqrt{\pi}}[-Ei(-2ab)]$$

4. $\int_0^\infty \operatorname{erfc}\left(\frac{b}{x}\right)\operatorname{erfc}(ax)\,dx$

$$= \int_0^\infty \operatorname{erfc}(bx)\operatorname{erfc}\left(\frac{a}{x}\right)\frac{dx}{x^2} = \frac{1}{a\sqrt{\pi}}e^{-2ab} + \frac{2b}{\sqrt{\pi}}[-Ei(-2ab)]$$

5. $\int_0^\infty \operatorname{erfc}(ax)\operatorname{erf}(bx)e^{b^2x^2}dx = \frac{-1}{2b\sqrt{\pi}}\ln\left(1-\frac{b^2}{a^2}\right), \qquad a^2 > b^2$

6. $\int_0^\infty \operatorname{erfc}(ax)\operatorname{erfc}(bx)e^{b^2x^2}dx = \frac{1}{b\sqrt{\pi}}\ln\left(1+\frac{b}{a}\right), \qquad a+b>0$

7. $\int_0^1 \operatorname{erf}(x)\operatorname{erf}(\sqrt{1-x^2})xdx = \frac{1}{4}\left(\frac{3}{e}-1\right)$

8. $\int_0^\infty \operatorname{erfc}(bx)\left[(x^2+a^2)\operatorname{erfc}\left(\frac{a}{x\sqrt{2}}\right)-\frac{2}{\sqrt{\pi}}axe^{-\frac{a^2}{2x^2}}\right]dx$

$$= \frac{1}{3b^3\sqrt{\pi}}\{e^{-2ab}(2a^2b^2-ab+1)-[-Ei(-2ab)]\}$$

9. $\int_0^\infty \operatorname{erfc}(cx)e^{a^2x^2}\left[2\cosh ab - e^{-ab}\operatorname{erf}\left(\frac{b-2ax^2}{2x}\right)-e^{ab}\operatorname{erf}\left(\frac{b+2ax^2}{2x}\right)\right]dx$

$$= \frac{1}{a\sqrt{\pi}}\{e^{-ab}[-Ei(-bc+ba)]-e^{ab}[-Ei(-bc-ba)]\}.$$

4.8. Combination of Error Function With Bessel Functions

1. $\int_0^\infty \operatorname{erf}(ax)J_0(bx)dx = \frac{1}{b}\operatorname{erfc}\left(\frac{b}{2a}\right)$

2. $\int_0^\infty \operatorname{erfc}(ax)J_0(bx)dx = \frac{1}{b}\operatorname{erf}\left(\frac{b}{2a}\right)$

3. $$\int_0^\infty \left[2\ \mathrm{erfc}\ (x)-1\right] J_0(bx)dx=-\frac{1}{b}\left[2\ \mathrm{erfc}\left(\frac{b}{2}\right)-1\right]$$

4. $$\int_0^\infty \mathrm{erf}\ (x)J_1(bx)dx=\frac{1}{b}\,e^{-b^2/8}I_0\left(\frac{b^2}{8}\right)$$

5. $$\int_0^\infty \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)J_{3/2}(bx)x^{-1/2}dx=b^{-3/2}\ \mathrm{erf}\left(\frac{b}{\sqrt{2}}\right)$$

6. $$\int_0^\infty \mathrm{erf}\ (ax)J_p(bx)x^p dx=\frac{1}{\sqrt{\pi}}\left(\frac{2}{b}\right)^p\frac{1}{b}\,\Gamma\left(p+\frac{1}{2},\frac{b^2}{4a^2}\right),\qquad -1<p<\frac{1}{2}$$

7. $$\int_0^\infty \mathrm{erfc}\ (ax)J_p(bx)x^p dx=\frac{1}{\sqrt{\pi}}\left(\frac{2}{b}\right)^p\frac{1}{b}\,\gamma\left(p+\frac{1}{2},\frac{b^2}{4a^2}\right),\qquad p>-1$$

8. $$\int_0^\infty \mathrm{erfc}\ (ax)J_p(bx)x^{p+1}dx=\frac{1}{2\sqrt{\pi}}\left(\frac{b}{2}\right)^p\frac{1}{a^{2p+2}}\frac{\Gamma\left(p+\frac{3}{2}\right)}{\Gamma(p+2)}\,{}_1F_1\left(p+\frac{3}{2};p+2;-\frac{b^2}{4a^2}\right),\qquad p>-1$$

9. $$\int_0^\infty \mathrm{erf}\ (ax)J_p(bx)x^{1-p}dx=\frac{1}{b}\left(\frac{b}{2}\right)^{p-1}\frac{1}{\Gamma(p)}\,{}_1F_1\left(\frac{1}{2};p;-\frac{b^2}{4a^2}\right),\qquad p>\frac{1}{2}$$

10. $$\int_0^\infty \mathrm{erfc}\ (ax)J_\nu(bx)x^p dx=\frac{2^{-(p+2\nu+1)}b^\nu\Gamma(p+\nu+1)}{a^{p+\nu+1}\Gamma(\nu+1)\Gamma\left(\frac{p+\nu+3}{2}\right)}$$
$$\times\,{}_2F_2\left(\frac{p+\nu+1}{2},\frac{p+\nu+2}{2};\nu+1,\frac{p+\nu+3}{2};-\frac{b^2}{4a^2}\right),\qquad p+\nu>-1$$

11. $$\int_0^\infty \mathrm{erfc}\ (ax)J_0(bx)e^{a^2x^2}x dx=\frac{1}{ab\sqrt{\pi}}\left[1-\sqrt{\pi}\left(\frac{b}{2a}\right)e^{b^2/4a^2}\ \mathrm{erfc}\left(\frac{b}{2a}\right)\right]$$

12. $$\int_0^\infty \mathrm{erfc}\ (ax)J_p(bx)e^{a^2x^2}x^{p+1}dx=\frac{1}{2\pi}\left(\frac{b}{2}\right)^p\frac{1}{a^{2p+2}}\,\Gamma\left(p+\frac{3}{2}\right)e^{b^2/4a^2}\Gamma\left(-p-1,\frac{b^2}{4a^2}\right),$$
$$-1<p<\frac{1}{2}$$

13. $$\int_0^\infty \mathrm{erfc}\ (ax)J_p(bx)e^{a^2x^2}x^p dx=\frac{b^{p+1/2}\Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}a^{3p/2+1}2^{p+1}}\,U\left(p+\frac{1}{2},p+1,\frac{b^2}{4a^2}\right)$$
$$=\frac{1}{\sqrt{\pi}}\frac{1}{ba^p}\,\Gamma\left(p+\frac{1}{2}\right)e^{b^2/8a^2}W_{-p/2,\,p/2}\left(\frac{b^2}{8a^2}\right),\qquad -1<p<\frac{3}{2}$$

14. $$\int_0^\infty \mathrm{erfc}\ (ax)J_p(bx)e^{a^2x^2}x^{p+1}dx=\frac{1}{2\pi}\left(\frac{b}{2}\right)^p\frac{1}{a^{2p+2}}\,\Gamma\left(p+\frac{3}{2}\right)e^{b^2/4a^2}\Gamma\left(-p-\frac{1}{2},\frac{b^2}{4a^2}\right),$$
$$-1<p<\frac{1}{2}$$

15. $$\int_0^\infty \mathrm{erfc}\ (x)Y_p(bx)e^{x^2}x^{p+1}dx=\frac{1}{2\pi}\left(\frac{b}{2}\right)^p\Gamma(p+1)e^{b^2/4}\Gamma\left(-p,\frac{b^2}{4}\right),\qquad -1<p<\frac{1}{2}$$

16. $$\int_0^\infty \mathrm{erfc}\ (x)Y_p(bx)e^{x^2}x^{p+3}dx=-\frac{1}{b\pi}\,\Gamma(p+2)e^{b^2/8}W_{-(p+3)/2,\,p/2}\left(\frac{b^2}{4}\right),\qquad -2<p<-\frac{3}{2}$$

17. $\int_0^\infty \operatorname{erfc}(x) I_p\left(\frac{1}{2}x^2\right) e^{x^2/2} x^{2p+1} dx = \frac{\Gamma\left(2p+\frac{3}{2}\right)\Gamma(-p)}{2\pi^{3/2}\left(p+\frac{1}{2}\right)}, \qquad -\frac{1}{2} < p < 0$

18. $\int_0^\infty \operatorname{erfc}(x) I_p\left(\frac{1}{2}x^2\right) e^{x^2/2} x^{2p} dx = \frac{\Gamma\left(2p+\frac{1}{2}\right)}{2\Gamma(p+1)\cos p\pi}, \qquad -\frac{1}{4} < p < \frac{1}{2}$

19. $\int_0^\infty \operatorname{erfc}(ax) I_p(x^2) e^{-(1-a^2)x^2} x^{2p+1} dx$

$$= \frac{\Gamma\left(2p+\frac{3}{2}\right)\Gamma(-p)}{\pi\left(p+\frac{1}{2}\right)} 2^{p-2} a^{-4p-2} {}_2F_1\left(p+\frac{1}{2}, 2p+\frac{3}{2}; p+\frac{3}{2}; 1-\frac{2}{a^2}\right),$$

$$p \neq -\frac{1}{2}, \qquad \mathscr{R}(a^2) > 1, \qquad -1 < \mathscr{R}(p) < 0$$

20. $\int_0^\infty \operatorname{erfc}\left(\frac{a}{\sqrt{2x}}\right) K_p(x) e^{a^2/2x - x} \frac{dx}{x} = \frac{\pi^{5/2}}{4} \sec p\pi \{[J_p(a)]^2 + [Y_p(a)]^2\}, \qquad -\frac{1}{2} < p < \frac{1}{2}$

21. $\int_0^\infty \operatorname{erfc}(x) J_{\lambda+\nu}(ax) J_{\lambda-\nu}(ax) x^p dx$

$$= \frac{a^{2\lambda}\Gamma\left(\lambda+\frac{1}{2}p+1\right) {}_4F_4\left(1+\lambda, \frac{1}{2}+\lambda, 1+\lambda+\frac{p}{2}, \frac{1}{2}+\lambda+\frac{p}{2}; 1+\lambda+\nu, 1+\lambda-\nu, 1+2\lambda, \frac{3}{2}+\lambda+\frac{p}{2}; -a^2\right)}{\sqrt{\pi} 2^{2\lambda+1}\Gamma(\lambda+\nu+1)\Gamma(\lambda-\nu+1)\left(\lambda+\frac{1}{2}p+\frac{1}{2}\right)},$$

$$\lambda + \frac{1}{2}p > 0, \qquad 1 + 2\lambda \neq -n.$$

4.9. Combination of Error Function With Other Special Functions

1. $\int_0^\infty \operatorname{erfc}(ax)\left[-Ei\left(-\frac{1}{4}x^2\right)\right]\frac{dx}{x} = (\gamma + \ln a)^2 + \zeta(2) + 2\sum_{k=0}^{\infty} \frac{(-a)^{k+1}}{k!(k+1)^3}$

2. $\int_0^\infty \operatorname{erfc}(ax)\left[-Ei\left(-\frac{b^2}{x^2}\right)\right]\frac{dx}{x^3} = \frac{1}{2b^2}(1-2ab)e^{-2ab} + 2a^2[-Ei(-2ab)]$

3. $\int_0^\infty \operatorname{erfc}(x) si(2px) dx = (e^{-1/4a^2} - 1) - \frac{\sqrt{\pi}}{2a}\operatorname{erfc}\left(\frac{1}{2a}\right)$

4. $\int_0^\infty \operatorname{erf}(ax) Ci(x) \frac{dx}{x} = -\frac{1}{8}[\zeta(2) + (\gamma - \ln 4a^2)^2] - \frac{1}{4}\sum_{k=0}^{\infty}\left(-\frac{1}{4a^2}\right)^{k+1} \frac{1}{k!(k+1)^3}$

5. $\int_1^\infty \operatorname{erfc}(ax) P_\nu^\mu(x) e^{a^2x^2}(x^2-1)^{-\mu/2} dx$

$$= \frac{2^{\mu-1}}{\pi} a^{\mu-3/2} e^{a^2/4} \Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}\right) W_{(1-2\mu)/4, (1+2\mu)/4}(a^2),$$

$$\mu < 1, \qquad \mu < \nu; \qquad \mu + \nu > -1, \qquad \mu - \nu \neq -2n$$

6. $$\int_0^\infty \operatorname{erfc}(x) L_\nu^{(p)}(x^2) x^{2p+1} dx = \frac{\Gamma\left(p+\frac{3}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{2\pi\nu!(p+\nu+1)}, \qquad p>-1$$

7. $$\int_0^\infty \operatorname{erfc}(x) L_\nu^{(p)}(ax^2) e^{-(a-1)x^2} x^{2p+1} dx$$

$$= \frac{\Gamma\left(p+\frac{3}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{2\pi(p+\nu+1)\Gamma(\nu+1)} {}_2F_1(p+\nu+1, p+\frac{3}{2}; p+\nu+2; 1-a), \qquad |1-a|<1, \qquad p>-1$$

8. $$\int_0^\infty \operatorname{erfc}(x) {}_1F_1\left(b-a+\frac{1}{2}; 2b+1; x^2\right) x^{4b} dx = \frac{\Gamma\left(4b+1\right)\Gamma\left(a-b+\frac{1}{2}\right)}{2^{4b+1}\Gamma(a+b+1)}$$

$$\mathscr{R}(b)>-\frac{1}{4}, \mathscr{R}(b-a)<\frac{1}{2}.$$

9. $$\int_0^\infty \operatorname{erf}(px) {}_1F_1(a; b; -p^2x^2) x^{2(b-1)} dx = \frac{\Gamma(b)}{\sqrt{\pi}p^{2b-1}(2b-1)} {}_2F_1(a, b-\frac{1}{2}; b+\frac{1}{2}; -1), \qquad b>\frac{1}{2}, \ a<\frac{1}{2}$$

10. $$\int_0^\infty \operatorname{erf}(px) {}_1F_1(a; b; -q^2x^2) x^{2(b-1)} dx = \frac{\Gamma(b)}{\sqrt{\pi}p^{2b-1}(2b-1)} {}_2F_1(a, b-\frac{1}{2}; b+\frac{1}{2}; \frac{-q^2}{p^2}), \qquad b>\frac{1}{2}, \ p2>q2$$

11. $$\int_0^\infty \operatorname{erf}(px) {}_1F_1(a; b; -q^2x^2) x^\nu dx$$

$$= \frac{1}{p^{\nu+1}\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2}\nu+1\right)}{(\nu+1)} {}_3F_2\left(a, \frac{\nu+2}{2}, \frac{\nu+1}{2}; b, \frac{\nu+3}{2}; \frac{-q^2}{p^2}\right), \qquad p\neq 0, \ \nu>-1, \qquad p^2>q^2$$

12. $$\int_0^\infty \operatorname{erf}(x) \Psi\left(a-\frac{1}{2}; b-\frac{1}{2}; px^2\right) x^{2b-2} dx$$

$$= \frac{1}{2p^b} \frac{\Gamma(b)}{(a-b)\Gamma(a)} {}_2F_1\left(\frac{1}{2}, b; a; 1-\frac{1}{p}\right), \qquad \mathscr{R}(p)\geqslant\frac{1}{2}, a\neq b$$

13. $$\int_0^\infty \operatorname{erf}(x) {}_1F_1\left(a; \frac{3}{2}\ qx^2\right) e^{-px^2} x dx$$

$$= \frac{(p+1)^{a-1/2}}{2p(p+1-q)^a} {}_2F_1\left[1, a; \frac{3}{2}; \frac{q}{p(p+1-q)}\right], \qquad p\neq 0, p+1\neq q, \mathscr{R}(p)>\mathscr{R}(q)$$

14. $$\int_0^\infty \operatorname{erfc}(x) {}_1F_1(a; b; -px^2) e^{x^2} x^{2b-1} dx = \frac{\Gamma(2b)\Gamma\left(a-b+\frac{1}{2}\right)}{\sqrt{\pi}\ 2^{2b}\Gamma(a+1)} {}_2F_1\left(a; b+\frac{1}{2}; a+1; 1-p\right),$$

$$\mathscr{R}(b)>0, \qquad \mathscr{R}(b-a)<\frac{1}{2}, \qquad |1-p|<1$$

15. $$\int_0^1 \operatorname{erf}(x) {}_1F_1(a; b; 1-x^2) e^{x^2} (1-x^2)^{b-1} x dx = \frac{\Gamma(b)}{2\Gamma\left(b+\frac{3}{2}\right)} {}_1F_1\left(a+1; b+\frac{3}{2}; 1\right),$$

$$\mathscr{R}(b)>0$$

327-016 O-69—2

16. $$\int_0^q \operatorname{erf}(ax)\,{}_1F_1\left[p+\frac{1}{2};\ p;\ b(q^2-x^2)\right]e^{a^2x^2}(q^2-x^2)^{p-1}x\,dx$$

$$=\frac{a}{2}\,q^{2p+1}e^{a^2q^2}\,\frac{\Gamma(p)}{\Gamma\left(p+\frac{3}{2}\right)}\,{}_1F_1[1;\ p+\tfrac{3}{2};\ q^2(b-a^2)],\qquad p\geq 1$$

17. $$\int_0^\infty \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right)D_\nu(\pm x)e^{x^2/4}dx=\frac{2^{(\nu+1)/2}}{\nu+1}\left[\frac{1}{\Gamma\left(\frac{1-\nu}{2}\right)}\mp\frac{\sqrt{\pi}}{\Gamma\left(-\frac{\nu}{2}\right)}\right],$$

$$\Re(\nu)\neq -1;\qquad \Re(\nu)>-1\quad\text{for lower sign}$$

18. $$\int_0^\infty \operatorname{erfc}\left(\frac{a}{\sqrt{2}}x\right)[D_\nu(x)-D_\nu(-x)]\,e^{(2a-1)x^2/4}x\,dx$$

$$=\frac{2^{\frac{\nu}{2}+2}a^{-3/2}}{(\nu\pi)}\,{}_2F_1\left(\frac{1}{2}\nu+1,\ 2;\ \frac{1}{2}\nu+2;\ 1-\frac{1}{a}\right),$$

$$\Re(a)>\frac{1}{2},\qquad \Re(\nu)>0$$

19. $$\int_0^\infty \operatorname{erfc}(x)M_{\mu,\nu}(ax^2)\,e^{(2-a)x^2/2}x^{2\nu-1}dx$$

$$=\frac{\Gamma(4\nu+1)\Gamma(\mu-\nu+\frac{1}{2})\ a^{\nu+1/2}}{\Gamma(\mu+\nu+1)2^{4\nu+1}}\,{}_2F_1\left(\mu+\nu+\frac{1}{2},\ 2\nu+\frac{1}{2};\ \mu+\nu+1;\ 1-a\right),$$

$$|a-1|<1,\qquad \Re(\nu)>-\frac{1}{4},\qquad \Re(\nu-\mu)<\frac{1}{2}.$$

20. $$\int_0^\infty \operatorname{erfc}(x)M_{\mu,\nu}(ax^2)e^{-1/2(a-2)x^2}x^{2\nu}dx$$

$$=\frac{\Gamma(2\nu+1)\Gamma\left(2\nu+\frac{3}{2}\right)\Gamma(\mu-\nu)p^{\nu+1/2}}{2\pi\Gamma\left(\mu+\nu+\frac{3}{2}\right)}\,{}_2F_1\left(\mu+\nu+\frac{1}{2},\ \frac{3}{2}+2\nu;\ \mu+\nu+\frac{3}{2};\ 1-a\right),$$

$$|a-1|<1,\qquad \Re(\mu)>\qquad \Re(\nu)>-\frac{1}{2}$$

21. $$\int_0^\infty \operatorname{erfc}(x)\,M_{\lambda,\mu}(ax^2)x^p\exp(ax^2/2)\,dx$$

$$=\frac{\Gamma\left(\mu+\frac{1}{2}p+\frac{3}{2}\right)a^{\mu+1/2}}{2\sqrt{\pi}\left(\mu+\frac{1}{2}p+1\right)}\,{}_3F_2\left(\lambda+\mu+\frac{1}{2},\ \mu+\frac{p}{2}+\frac{3}{2},\ \mu+\frac{p}{2}+1;\ 2\mu+1,\ \mu+\frac{p}{2}+2,\ -a\right),$$

$$\mu+\frac{1}{2}p+1>0.$$

5. Appendix. Some Relevant Integrals Involving Elementary Functions

(A1) $$\int z^n e^{az}dz=e^{az}\sum_{k=0}^{n}(-1)^k\,\frac{n!}{(n-k)!}\,\frac{z^{n-k}}{a^{k+1}}$$

(A2) $$\int e^{-a^2z^2+bz}dz=\frac{\sqrt{\pi}}{2a}\exp\left(\frac{b^2}{4a^2}\right)\operatorname{erf}\left(az-\frac{b}{2a}\right)$$

(A3) $$\int_0^\infty e^{-a^2x^2+bx}dx=\frac{\sqrt{\pi}}{2a}\exp\left(\frac{b^2}{4a^2}\right)\left[1+\operatorname{erf}\left(\frac{b}{2a}\right)\right]$$

(A4) $$\int ze^{-a^2z^2+bz}dz=\frac{1}{2a^2}\exp\left(\frac{b^2}{4a^2}\right)\left[\frac{b\sqrt{\pi}}{2a}\operatorname{erf}\left(az-\frac{b}{2a}\right)-e^{-(az-b/2a)^2}\right]$$

(A5) $$\int_0^\infty xe^{-a^2x^2+bx}dx=\frac{1}{2a^2}\left[\frac{\sqrt{\pi}\,b}{2a}e^{b^2/4a^2}\operatorname{erfc}\left(-\frac{b}{2a}\right)+1\right]$$

(A6) $$\int_0^\infty x^ne^{-a^2x^2+bx}dx=(2a^2)^{-(n+1)/2}n!\exp\left(\frac{b^2}{8a^2}\right)D_{-(n+1)}\left(-\frac{b}{a\sqrt{2}}\right)$$

(A7) $$\int z^ne^{-a^2z^2+bz}dz=a^{-n-1}\exp\left(\frac{b^2}{4a^2}\right)\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\left(\frac{b}{2a}\right)^{n-k}\int u^ke^{-u^2}du,\qquad u=az-\frac{b}{2a}$$

(A8) $$\int u^ke^{-u^2}du=-\frac{e^{-u^2}}{2}\sum_{j=0}^{r-1}\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}-j\right)}u^{k-2j-1}+\frac{1}{2}(1-s)\Gamma\left(r+\frac{1}{2}\right)\operatorname{erf}(u),$$

$$k=2r-s,\qquad s=0\text{ or }1$$

(A9) $$(n-1)\int z^{-n}e^{-a^2z^2}dz=-z^{-n+1}e^{-a^2z^2}-2a^2\int z^{-n+2}e^{-a^2z^2}dz$$

(A10) $$\int z^{-n}e^{-a^2z^2}dz=\frac{e^{-a^2z^2}}{2a^2\Gamma\left(\frac{n+1}{2}\right)}\sum_{k=1}^{n/2}(-1)^k\Gamma\left(\frac{n+1}{2}-k\right)a^{2k}z^{2k-n-1}$$

$$+\frac{(-1)^{n/2}\pi a^{n-1}}{2\Gamma\left(\frac{n+1}{2}\right)}\operatorname{erf}(az),\qquad n\text{ even positive integer}$$

$$=\frac{e^{-a^2z^2}}{2a^2\Gamma\left(\frac{n+1}{2}\right)}\sum_{k=1}^{(n-1)/2}(-1)^k\Gamma\left(\frac{n+1}{2}-k\right)a^{2k}z^{2k-n-1}$$

$$+\frac{(-1)^{(n-1)/2}}{2\left(\frac{n-1}{2}\right)!}Ei(-a^2z^2),\qquad n>1\text{ odd positive integer}$$

(A11) $$\int f(z)e^{-(a^2z^2+2bz)}dz=\frac{1}{a}e^{b^2/a^2}\int f\left(\frac{au-b}{a^2}\right)e^{-u^2}du,\qquad\text{where } u=a\left(z+\frac{b}{a^2}\right)$$

(A12) $$\int(z-a)^ne^{-c^2(z-b)^2}dz=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{(b-a)^{n-k}}{c^{k+1}}\int u^ke^{-u^2}du,\qquad\text{where } u=c(z-b)$$

(A13) $$B_1(p,\alpha,u)\equiv\int u^pe^{-\alpha u}\ln u\,du=-u^{p+1}\left[\sum_{j=0}^{\infty}\frac{(-\alpha u)^j}{j!(p+j+1)^2}-\ln u\sum_{j=0}^{\infty}\frac{(-\alpha u)^j}{j!(p+j+1)}\right],\qquad p>-1$$

(A14) $$\alpha B_1(p,\alpha,u)=pB_1(p-1,\alpha,u)+\frac{1}{\alpha^p}\gamma(p,\alpha u)$$

(A15) $$\int_u^\infty\ln xe^{-ax}\frac{dx}{x}=\frac{1}{2}\{(\gamma+\ln au)^2+\zeta(2)+2\ln uE_1(au)\}+\sum_{k=0}^{\infty}\frac{(-au)^{k+1}}{k!(k+1)^3}$$

(A16) $$\int_0^\infty x^pe^{-ax}\ln x\,dx=\frac{\Gamma(p+1)}{a^{p+1}}[\psi(p+1)-\ln a]$$

6. References

[1] Bhattacharya, S. K., Proc. Camb. Phil. Soc. **63,** 179 (1967).

[2] Bierens de Haan, D., Nouvelles Tables d'Intégrales Définies. (New York, G. E. Stechert and Company, 1939.)

[3] Bock, P., Composito. Mathematica J. **7,** 123, (1939).

[4] Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G., Higher Transcendental Functions, Vol. **1** and **2** (McGraw-Hill Book Co., Inc., New York 1953). Tables of Integral Transforms. Vols. **1** and **2** (McGraw-Hill Book Co., Inc., New York 1954).

[5] Gautschi, W., Chapter 7 of [9] below.

[6] Geller, M., A Table of Integrals Involving Powers, Exponentials, Logarithms and the Exponential Integral. Jet Propulsion Laboratory Technical Report No. 32-469(1963).

[7] Gradshteyn, I. S., and Ryzhik, I. W., Table of Integrals, Series and Products. (Academic Press, New York 1965), pages 648-653.

[8] Gröbner, W., and Hofreiter, N., Integraltafeln (Springer-Verlag, New York 1965).

[9] Abramowitz, M. and Stegun, I. A. (Editors), Handbook of Mathematical Functions, NBS AMS **55,** (Government Printing Office, Washington, D. C. 1965).

[10] Jahnke, E., Emde, F., and Lösch, F., Tables of Higher Functions (McGraw-Hill Book Co., Inc., New York 1960).

[11] Magnus, W., Oberhettinger, F., and Soni, R. P., Special Functions of Mathematical Physics (Springer-Verlag, New York 1966).

[12] Mangulis, V., Handbook of Series (Academic Press, New York 1965).

[13] Ng, E. W., Astrophysical J. **150,** 787 (1967).

[14] Short, P. J., Rev. Soc. Indust. Appl. Math. **7,** 565 (1965).

[15] Zelen, M. and Severo, N. C., Chapter 26 of [9] above.

[16] Zimering, S., J. Mathematical Phys. **6,** 336 (1965).

(Paper 73B1-281)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 73B, No. 1, January–March 1969

Fourier Coefficients of Mathieu Functions in Stable Regions*

Henning Früchting**

(October 25, 1968)

A method for calculating the Fourier coefficients of Mathieu functions in stable regions based on the method of bisection and Miller's recurrence algorithm is introduced. Some Fourier coefficients are calculated and compared with those given earlier by Tamir and Wang. It is shown that the method of Tamir and Wang fails for indices in the neighborhood of integers.

Key Words: Bisection method; eigenvalues; Fourier coefficients; Mathieu functions.

It is known from the theory of Mathieu functions [1][1] that the Floquet solution of the differential equation

$$y''(z) + (\lambda - 2h^2 \cos 2z)y(z) = 0 \tag{1}$$

with characteristic exponent ν has the Fourier expansion

$$me_\nu(z, h^2) = \sum_{n=-\infty}^{\infty} c_{2n}^{\nu}(h^2)\, e^{i(\nu+2n)z}. \tag{2}$$

The Fourier coefficients satisfy the following three-term recurrence formula:

$$-h^2 c_{2n+2}^{\nu}(h^2) + [\lambda - (\nu + 2n)^2]c_{2n}^{\nu}(h^2) - h^2 c_{2n-2}^{\nu}(h^2) = 0. \tag{3}$$

Continued fraction expansions for the eigenvalues $\lambda = \lambda(\nu, h^2)$ and for the Fourier coefficients c_{2n}^{ν}/c_0^{ν} are given by Tamir and Wang in [2]. These relations are valid both in the stable (ν real) and unstable (ν complex) regions. $F(\nu) \equiv c_{2n}^{\nu}/c_0^{\nu}$ is numerically evaluated and depicted. It is seen from the graphs in [2] that the given formulas for c_{2n}^{ν}/c_0^{ν} are not particularly suitable for numerical evaluation in the stable regions. Representations for $F(\nu) = c_{2n}^{\nu}/c_0^{\nu}$ derived from other numerical methods will now be introduced.

The eigenvalues λ of Mathieu's differential equation can be determined as the eigenvalues of the symmetrical tridiagonal matrix of the homogeneous system of eq (3) with the help of the bisection method [3]. Then the Fourier coefficients are obtainable from the recurrence formula (3) under consideration of stability (Miller's recurrence algorithm [4, 5]). The accuracy of this method depends principally upon the number of equations used.

Numerical work was carried out to single precision on the computer of the Technische Hochschule Darmstadt. For evaluating the eigenvalues λ, a modified version of the bisection method [6] was used. This modification prohibits overflow. Also, a numerical test ensures that the trunca-

[1] Figures in brackets indicate the literature references at the end of this paper.
*An invited paper.
**Present address: Lehrstuhl für Theoretische Elektrotechnik, Technische Hochschule, 61 Darmstadt, Schlossgartenstrasse 2, Germany.

tion error is smaller than the rounding error of the bisection method, when the number of equations used is $[\nu+h^2]+8$. The equations are set up in such a manner that with increasing n the equation with the least value of $|\nu+2n|$ is in the middle.

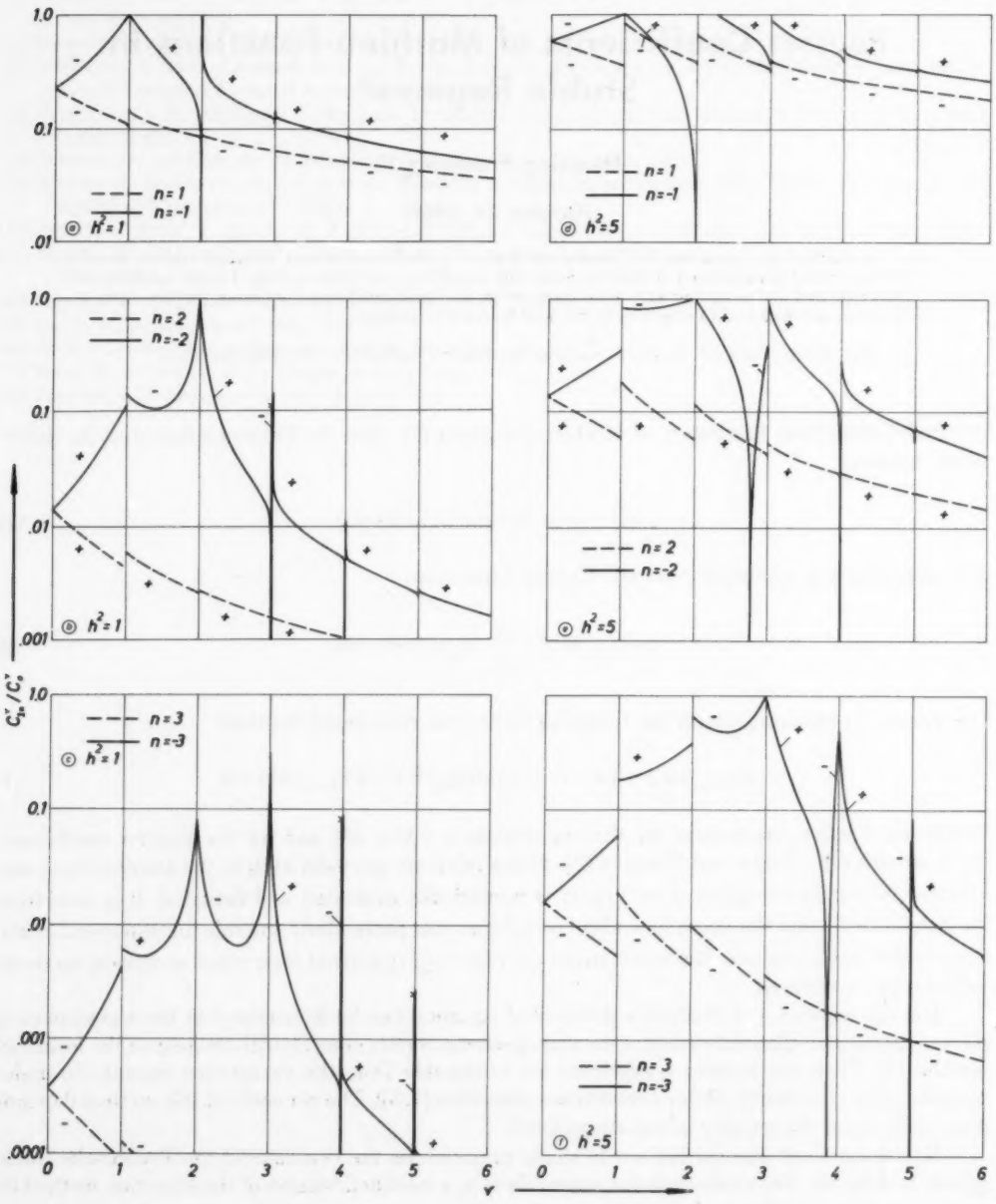

FIGURE 1: *Modulus and sign of* $c_{2n}^{\nu}(h^2)/c_0^{\nu}(h^2)$ *as a function of* ν *for two values of the parameter* h^2.

To avoid overflow in applying (3), this relation is rewritten as a recurrence formula for the quotients $c^{\nu}_{2n}/c^{\nu}_{2n-2}$, given by

$$\frac{c^{\nu}_{2n+2}}{c^{\nu}_{2n}} - \frac{\lambda-(\nu+2n)^2}{h^2} + \left(\frac{c^{\nu}_{2n}}{c^{\nu}_{2n-2}}\right)^{-1} = 0; \tag{4}$$

compare [7]. The starting values $c^{\nu}_{2N}/c^{\nu}_{2N-2}$ and $c^{\nu}_{-2N}/c^{\nu}_{-2N+2}$ were evaluated from continued fraction expansions obtained from (4) with $c^{\nu}_{2(N+m)}$ and $c^{\nu}_{-2(N+m)}$ set equal to zero. A numerical test determines the number of elements, m, to be used.

Figure 1 shows that as a function of ν, c^{ν}_{2n}/c^{ν}_{0} is continuous for noninteger ν. It has n zeros; they lie near integer values of ν for small h^2 (fig. 1a–c). For increasing values of h^2 the zeros move toward smaller ν (fig. 1d–f). For interger ν, however, c^{ν}_{2n}/c^{ν}_{0} is discontinuous. The relation

$$c^{m}_{r-m}(h^2) = c^{m}_{-r-m}(h^2) \tag{5}$$

in which r is an arbitrary integer, is valid for the limiting value from the right, giving $\lambda = a_m$, say, and $\nu = m$, $m = 0, 1, \ldots$. Similarly,

$$c^{m}_{r-m}(h^2) = -c^{m}_{-r-m}(h^2) \tag{6}$$

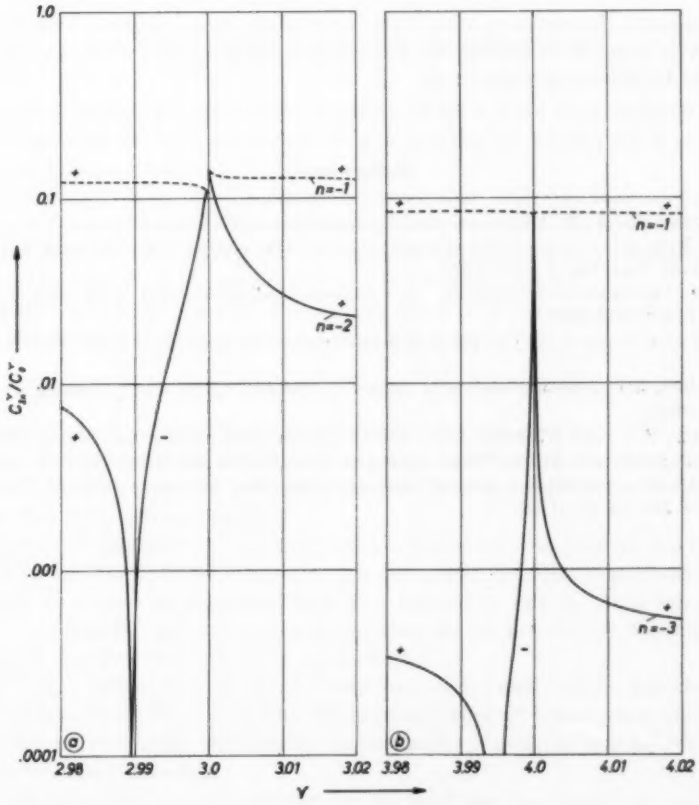

FIGURE 2: *Modulus and sign of* $c^{\nu}_{2n}(1)/c^{\nu}_{0}(1)$ *in the region* $\nu \approx$ m.
a. m = 3, illustrating c^{ν}_{-3}/c^{ν}_{3} and c^{ν}_{-4}/c^{ν}_{4} b. m = 4, illustrating c^{ν}_{-3}/c^{ν}_{3} and c^{ν}_{-4}/c^{ν}_{4}

is valid for the limiting value from the left, giving $\lambda = b_m$, say, and $\nu = m$, $m = 1, 2, \ldots$; compare [1], p. 116, and [2]. In particular, for even values of m we have from (6)

$$c^m_{-m}(h^2) = -c^m_{-m}(h^2) = 0. \tag{6a}$$

Fourier coefficients with other indices coinciding according to eqs (5) and (6) with $\nu \to m$ are shown in figure 2. The absolute values of the Fourier coefficients with negative indices vary rapidly in the region $m - \epsilon < \nu < m$, $\epsilon \ll 1$.

Tamir and Wang [2] calculated only the zeros of c^ν_{2n}/c^ν_0 which result from eq (6a), although by use of (5) and (6) it is also seen that for $\nu \neq m$ c^ν_{2n}/c^ν_0 can vanish. Thus the graphs in [2] become false. This leads to an erroneous calculation of Mathieu functions with indices in the neighborhood of integers.

Summary

A method for calculating the Fourier coefficients of Mathieu functions in stable regions based on the method of bisection and Miller's recurrence algorithm is introduced. Some Fourier coefficients are calculated and compared with those given earlier by Tamir and Wang. It is shown that the method of Tamir and Wang fails for indices in the neighborhood of integers.

The author is grateful to Dr.-Ing. G. Piefke for suggesting this work, which was sponsored by the Deutsche Forschungsgemeinschaft.

References

[1] Meixner, J., and Schäfke, F. W., Mathieusche Funktionen und Sphäroidfunktionen (Springer-Verlag, Berlin, 1954).

[2] Tamir, T., and Wang, H. C., Characteristic relations for nonperiodic solutions of Mathieu's equation, J. Res. NBS **69B** (Math. and Math. Phys.) Nos. 1 and 2 (1965).

[3] Wilkinson, J. H., Calculation of the eigenvalues of a symmetric tridiagonal matrix by the method of bisection, Num. Mathematik **4**, 362–367 (1962).

[4] Abramowitz, M., and Stegun, I. A., Generation of Bessel functions on high speed computers, M.T.A.C. **11**, 255–257 (1957).

[5] Gautschi, W., Instability of linear second-order difference equations, Information Processing, Proc. IFIP Congress Munich, 207 (1963).

[6] Barth, W., Martin, R. S., and Wilkinson, J. H., Handbook series linear algebra. Calculation of the eigenvalues of a symmetric tridiagonal matrix by the method of bisection, Num. Mathematik **9**, 386–393 (1966–1967).

[7] Corbato, F. J., On the computation of auxiliary functions for two-center integrals by means of a high-speed computer, J. Chem. Phys. **24**, 452–453 (1956).

(Paper 73B1–282)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 73B, No. 1, January–March 1969

A Note on the T-Transformation of Lubkin*

W. D. Clark,** H. L. Gray,*** and J. E. Adams**

(October 25, 1968)

This paper is concerned with a sequence-to-sequence transformation studied extensively by Samuel Lubkin [J. Res. NBS 48, 228–254 (1952)]. Lubkin has studied the rate of convergence of the transformed sequence, $\{T_n\}$, versus the original sequence, $\{S_n\}$. In this respect, the authors have shown that a more accurate evaluation of the transformation is achieved by the comparison of $\{T_n\}$ with $\{S_{n+1}\}$ instead of $\{S_n\}$. The main theorems proved are rate-of-convergence comparisons between $\{T_n\}$ and $\{S_{n+1}\}$ where $\{S_n\}$ is the sequence of partial sums of a convergent series whose terms are of constant sign or else are alternating.

Key Words: Convergence acceleration techniques; epsilon-transformation; nonlinear series transformation; numerical methods; series summability methods.

The sequence-to-sequence T-transformation (defined later by (1.1)) introduced by Aitken [1][1] has been studied extensively by Lubkin [4] and more recently by Shanks [6], Wynn [7], Marx [5], Gray and Atchison [3] and others.

If a sequence, S_n, converges then Lubkin has shown that, under certain conditions, its image sequence, $T(S_n) = T_n$, converges to this same limit. Moreover, he has shown that, under certain conditions, the sequence T_n converges more rapidly than the sequence S_n. Consequently, this transform may be used to accelerate the convergence of some infinite series.

In applying this transform, however, the authors discovered a seeming paradox. The following theorem is stated and proved as Theorem 4 on page 231 of [3].

THEOREM A: *If* $S_n = \sum_{k=0}^{n} a_k$ *converges and if* $R_n = \frac{a_{n+1}}{a_n}$ *is of constant sign and converges to 0 then the sequence* $T(S_n) = T_n$ *converges more rapidly than the sequence* S_n.

The authors of this paper assumed that if a series satisfied the hypothesis of this theorem then the transform could be applied with good results. However, this proved to be an erroneous assumption and thus the seeming paradox.

It is the purpose of this paper to clarify the above mentioned irregularity and introduce some theorems that give a more valid evaluation of the transform. It is the contention of the authors that the sequence T_n should be compared with S_{n+1} instead of with S_n. Although the sequences S_n and S_{n+1} are "essentially" the same sequences, they do not necessarily converge at the same rate. It is this point that has evidently been overlooked.

Looking back at Theorem A, it should be noted that series satisfying the hypotheses are either series having terms of constant sign or else alternating series. In either case, since $R_n \to 0$, the sequence $|a_k|$ must be eventually decreasing. Consequently this paper will be concerned, in the main, with these two types of series.

The usual definitions for rates of convergence are used and are quoted for completeness.

---

*An invited paper.

**Present address: Stephen F. Austin State College, Nacogdoches, Texas 75961.

***Present address: Texas Technological College, Lubbock, Texas 79409.

[1] Figures in brackets indicate the literature references at the end of this paper.

If $a_n \to a$ and $b_n \to b$ then;

DEFINITION I: a_n *converges more rapidly than* b_n *if and only if* $\lim_{n\to\infty} \frac{a-a_n}{b-b_n} = 0$. *This may also be stated as* b_n *converges less rapidly than* a_n.

DEFINITION II: a_n *converges with the same order of rapidity as* b_n *if and only if there exist constants* A *and* B *such that* $0 < A < \left|\frac{a-a_n}{b-b_n}\right| < B$ *for* n *sufficiently large. Note that if* $\frac{a-a_n}{b-b_n} \to A \neq 0$ *then* a_n *converges with the same order of rapidity as* b_n.

The next three theorems are of utmost importance to this paper and also provide a "L'Hospital's Rule" for certain types of sequences and series.

THEOREM 1: *If* $a_n \to 0$, $b_n \to 0$ *and* b_n *is monotone and if* $\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}}$ *exists, finite or infinite, then* $\lim_{n\to\infty} \frac{a_n}{b_n}$ *exists and* $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}}$.

This is proved by Bromwich [2] for the case b_n decreasing. The proof for b_n increasing is analagous.

Throughout this paper S_n will denote the partial sums of the convergent series $\sum_{n=0}^{\infty} a_n$, $R_n = \frac{a_{n+1}}{a_n}$ and $R = \lim_{n\to\infty} R_n$ if this limit exists.

THEOREM 2: *If* a_n *is of constant sign,* $S_n \to S$, *and* $R_n \to R$ *then* $\frac{S-S_{n+1}}{S-S_n} \to R$.

$$\text{i.e., } \lim_{n\to\infty} \frac{\sum_{k=n+2}^{\infty} a_k}{\sum_{k=n+1}^{\infty} a_k} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

PROOF: Let $A_n = S - S_{n+1}$ and $B_n = S - S_n$ and apply Theorem 1 to these sequences. B_n is monotone since a_n is of constant sign.

NOTE: If $R = 0$ then S_{n+1} converges more rapidly than S_n.

THEOREM 3: *If* $a_n = (-1)^n c_n$ *and* $c_n > c_{n+1} > 0$ *and if* $R_n \to R \neq -1$ *then* $\frac{S-S_{n+1}}{S-S_n} \to R$. *Moreover, if* $R = -1$ *and* $\frac{1+R_{n+1}}{1+R_n} \to 1$ *then the above result holds.*

PROOF: In this case, the sequences S_{2n} and S_{2n+1} are monotone since $c_n > c_{n+1}$ and thus $S - S_{2n}$ and $S - S_{2n+1}$ are also monotone. Consider first the sequences $A_n = S - S_{2n+1}$ and $B_n = S - S_{2n}$. We have then that $\frac{A_n - A_{n-1}}{B_n - B_{n-1}} = R_{2n-1}\left[\frac{1+R_{2n}}{1+R_{2n-1}}\right]$ and this converges to R. Hence $\frac{S-S_{2n+1}}{S-S_{2n}} \to R$. Similarly, the odd subsequence of $\frac{S-S_{n+1}}{S-S_n}$ converges to R and hence the result.

We are now ready to consider the T-transform which is defined by:

$$T_n = \frac{S_n^2 - S_{n-1}S_{n+1}}{2S_n - S_{n-1} - S_{n+1}} \tag{1.1}$$

where S_n is any sequence for which the denominator is not zero.

In this paper S_n will be a sequence of partial sums and thus T_n may be written,

$$T_n = S_n + \frac{a_{n+1}}{1-R_n} = \frac{S_{n+1} - R_n S_n}{1-R_n}.$$

For completeness, we will state two theorems proved by Lubkin as Theorems 1 and 2 on pages 230–1 of [4].

THEOREM 4: *If* S_n *converges to* S *and if* T_n *converges then* T_n *converges to* S.

THEOREM 5: *If* S_n *converges and if there exists* k *such that* $|1-R_n|>k>0$ *for* n *sufficiently large then* T_n *converges. As a special case, we note that if* R *exists and* $R\neq 1$ *then* T_n *converges.*

Referring once again to Theorem A and to Theorems 2 and 3, we see that for series satisfying the hypotheses of the first theorem, S_{n+1} converges more rapidly than S_n. If perchance S_{n+1} and T_n converge with the same order of rapidity, or if T_n converges less rapidly than S_{n+1}, one would feel that the transform does not yield very good results—since S_{n+1} must be computed in order to compute T_n. Thus the authors feel that T_n should be compared with S_{n+1} from the outset, rather than with S_n. The remainder of the paper makes this comparison.

LEMMA 1: *If* $\frac{R_n}{R_{n+1}}\to A$, $|A|\neq 1$ (*A finite or infinite*) *then* $R_n\to 0$ *or else* $|R_n|\to\infty$. *Moreover if* $|A|<1$ *then* $|R_n|\to\infty$ *and if* $|A|>1$ *then* $R_n\to 0$.

LEMMA 2: *If* $S_n\to S$ *and* $\frac{R_n}{R_{n+1}}\to A$, $|A|\neq 1$ *then* $R_n\to 0$ *and from Lemma 1 it follows that* $|A|>1$.

THEOREM 6: *If* $R_n\to R\neq 1, 0$ *and* $\frac{S-S_{n+1}}{S-S_n}\to R$ *then* T_n *converges more rapidly than* S_{n+1}.

PROOF: By Theorem 5, T_n converges to S. Also

$$\frac{S-T_n}{S-S_{n+1}}=\frac{1}{1-R_n}\left[1-\frac{R_n(S-S_n)}{S-S_{n+1}}\right]$$

and this converges to 0 when $R\neq 0, 1$.

THEOREM 7: *If* a_n *is of constant sign,* $S_n\to S$ *and*

$$\frac{R_n}{R_{n+1}}\to A$$

then

(a) *If* $A=1$ *and there exists* k *such that* $|1-R_n|>k>0$ *for* n *sufficiently large then* T_n *converges more rapidly than* S_{n+1} *and*

(b) *If* $1<A<\infty$ *then* T_n *converges with the same order of rapidity as* S_{n+1} *and*

(c) *If* $A=\infty$ *then* T_n *converges less rapidly than* S_{n+1}.

PROOF: Let $A_n=S-T_n$ and $B_n=S-S_{n+1}$. Then $A_n\to 0$ and B_n converges monotonically to 0 since a_n is of constant sign. Also

$$\frac{A_n-A_{n-1}}{B_n-B_{n-1}}=\frac{\left[1-\frac{R_{n-1}}{R_n}\right]}{(1-R_n)(1-R_{n-1})}. \tag{1.2}$$

In the case of (a), the absolute value of (1.2) is bounded by $\frac{\left|1-\frac{R_{n-1}}{R_n}\right|}{k^2}$ which converges to 0 and thus (1.2) converges to $0=1-A$. In cases (b) and (c), it is seen by Lemma 1 that $R_n\to 0$ and (1.2) converges to $1-A$. Thus by Theorem 1, $\frac{A_n}{B_n}=\frac{S-T_n}{S-S_{n+1}}\to 1-A$ in each case and the result in each case then follows.

THEOREM 8: *If* $a_n=(-1)^n c_n$ *where* $c_n>c_{n+1}>0$[2] *and if*

$$S_n\to S \text{ and } \frac{R_n}{R_{n+1}}\to A$$

then

(a) *If* $A=1$ *and there exists* k *such that* $|1+R_n|>k>0$ *for* n *sufficiently large then* T_n *converges more rapidly than* S_{n+1} *and*

[2] This can be shown to hold without the hypothesis $c_n>c_{n+1}$.

(b) *If* $1 < A < \infty$ *then* T_n *converges with the same order of rapidity as* S_{n+1} *and*
(c) *If* $A = \infty$ *then* T_n *converges less rapidly than* S_{n+1}.

PROOF: This will be proved by showing that the even and odd subsequences of $\dfrac{S-T_n}{S-S_{n+1}}$ both converge to $1-A$ in each of the above cases. Let $A_n = S - T_{2n}$ and $B_n = S - S_{2n+1}$. Then $A_n \to 0$ and B_n decreases to 0 since $c_n > c_{n+1}$. Also,

$$\frac{A_n - A_{n-1}}{B_n - B_{n-1}} = \frac{a_{2n-1}\dfrac{R_{2n-1}-R_{2n-2}}{(1-R_{2n-1})(1-R_{2n-2})} + a_{2n}\dfrac{R_{2n}-R_{2n-1}}{(1-R_{2n})(1-R_{2n-1})}}{a_{2n}+a_{2n+1}}$$

$$= \frac{1-\dfrac{R_{2n-2}}{R_{2n-1}}}{(1-R_{2n-1})(1-R_{2n-2})(1+R_{2n})} + \frac{R_{2n}-R_{2n-1}}{(1-R_{2n})(1-R_{2n-1})(1+R_{2n})}. \tag{1.3}$$

Since $R_n < 0$ for all n it follows that $|1-R_n| > 1$. Moreover, $|R_n| < 1$. In case (a), the absolute value of (1.3) is bounded by

$$\frac{\left|1-\dfrac{R_{2n-2}}{R_{2n-1}}\right|}{k} + \frac{\left|1-\dfrac{R_{2n-1}}{R_{2n}}\right|}{k}$$

which converges to 0 and thus (1.3) converges to $0 = 1-A$. In cases (b) and (c), $R_n \to 0$ and (1.3) converges to $1-A$. A similar argument applies to the odd subsequence and the result in each case then follows.

The following example will illustrate the considerations of this paper.

EXAMPLE I: *Let*

$$a_n = \frac{1}{n!}, \qquad b_n = \frac{1}{n!2^{n^2}}$$

and

$$C_n = \frac{1}{n!2^{n^3}}.$$

Then

$$R_n = \frac{a_{n+1}}{a_n} = \frac{1}{n+1}, \qquad V_n = \frac{b_{n+1}}{b_n} = \frac{1}{(n+1)2^{2n+1}}$$

and

$$W_n = \frac{C_{n+1}}{C_n} = \frac{1}{(n+1)2^{3n^2+3n+1}}.$$

Also

$$\frac{R_n}{R_{n+1}} = \frac{n+2}{n+1} \to 1, \qquad \frac{V_n}{V_{n+1}} = \frac{(n+2)}{(n+1)}4 \to 4$$

and

$$\frac{W_n}{W_{n+1}} = \frac{(n+2)2^{6n+6}}{(n+1)} \to \infty.$$

All three of the series associated with these sequences satisfy the hypotheses of Theorem A and thus it was felt that the transform should have given good results. This is however not the case. Theorem 7 can be invoked here to give a more accurate evaluation of the transform. It is seen then that the first series can be transformed with suitable results while the last two cannot.

We are grateful to the referee for his most valuable comments and suggestions.

References

[1] Aitken, A. C., "On Bernoulli's numerical solution of algebraic equations," Proc. Roy. Soc. (Edinburgh), **46,** (1926.

[2] Bromwich, T. J., An Introduction to the Theory of Infinite Series, p. 413 (Macmillan & Co. Ltd. New York, Second Edition, 1926).

[3] Gray, H. L., and Atchison, T. A., "Non-linear transformations related to the evaluation of improper integrals," Siam J. of Numerical Analysis, **4,** No. 3 (Sept. 1967).

[4] Lubkin, Samuel, A method of summing infinite series, J. Res. NBS **48,** 228–254 (RP2310).

[5] Marx, I., Remark concerning a non-linear sequence-to-sequence transform, "J. Math and Phys," **42,** 1963).

[6] Shanks, D., Non-linear transformations of divergent and slowly convergent sequences, Dissertation, University of Maryland (1954).

[7] Wynn, P., On the Convergence and Stability of the Epsilon Algorithm, J. Siam Numerical Analysis, **3,** No. 1 (1966).

(Paper 73B1–283)

JOURNAL OF RESEARCH of the National Bureau of Standards–B. Mathematical Sciences
Vol. 73B, No. 1, January–March 1969

The Cylinder Problem in Thermoviscoelasticity*

Warren S. Edelstein**

October 25, 1968

Solutions are obtained for several axially symmetric plane strain problems involving a hollow circular viscoelastic cylinder. The cylinder is assumed to be subject to planar, axially symmetric body force and temperature fields. Displacement, traction, and mixed boundary conditions are considered.

Key Words: Cylinder; Neumann series; thermoviscoelasticity.

1. Introduction

In this paper we study a hollow thick-walled right circular cylinder composed of an isotropic homogeneous viscoelastic material whose response to temperature is thermorheologically simple.

We solve the axially symmetric plane strain problem for this cylinder subject to three types of boundary conditions:

(i) zero displacement at the inner and outer walls,

(ii) zero traction on both walls, and

(iii) zero traction inside, zero displacement outside.

The latter mixed problem was chosen to suggest the solid fuel core of a rocket. Our analysis can be easily modified to deal with the case of a solid cylinder or with nonvanishing boundary conditions.

Many problems in isothermal viscoelasticity are readily solved by means of Laplace transforms. This technique has also been applied with some success in thermoviscoelasticity, for example to sphere problems by Muki and Sternberg [1][1] and Sternberg and Gurtin [2], and to a contact problem by Graham [3].

In fact Valanis and Lianis [9] use Laplace transforms to obtain the solution of a thermoviscoelastic cylinder problem under the assumption, unnecessary in the present paper, of a constant hereditary Poisson ratio.

Our method is fundamentally different from the above. It is a generalization of that used in [4] to obtain existence for the displacement problem of isothermal viscoelasticity. The analysis in [4] led to a Neumann expansion representation for the viscoelastic displacement field which involved the inverse E^{-1} of an elastic operator E. Also, convergence of the expansion depended on a Schauder estimate for E. In order to assure existence of E^{-1} and of the estimate we were obliged in that paper to confine ourselves to the displacement problem. For the solution of the cylinder problem we obtain an analogous Neumann expansion. However, due to the extremely simple geometry of the present problem an explicit representation of E^{-1} is easily derived for a variety of boundary conditions. Thus, our solution is altogether explicit, and moreover, estimates which suffice for the convergence of the expansion are immediate.

Finally we remark that no attempt has been made to expound on the derivation of the constitutive equations set forth below. Thorough expositions of the theory of linear viscoelasticity and

*An invited paper. This research was supported by the Office of Naval Research, contract number N-0001467-A-0210-001, task number NR 041-348. Reproduction in whole or in part is permitted for any purpose of the United States Government.

**Present address: Department of Mathematics, Illinois Institute of Technology, Chicago, Ill. 60616.

[1] Figures in brackets indicate the literature references at the end of this paper.

thermorheologically simple materials are given in the papers [5, 6] of Gurtin and Sternberg. If one accepts our field equations, then our analysis can be followed without consultation of any of the above references.

We plan a sequel to this paper in which the effects of ablation are considered and numerical examples are given.

2. Statement of the Problems

Consider a hollow circular cylinder with inner radius a and outer radius b. If x_1, x_2, and x_3 are orthogonal Cartesian coordinates oriented in such a way that the x_3-axis coincides with the axis of the cylinder, then a set of cylindrical coordinates ρ, φ, z is defined by

$$x_1=\rho\cos\varphi, \qquad x_2=\rho\sin\varphi, \qquad x_3=z$$

where

$$0\leq\rho<\infty, \qquad 0\leq\varphi<2\pi, \qquad -\infty<z<\infty.$$

The cylinder is composed of isotropic homogeneous viscoelastic material subject to a temperature field[2] $T(\rho, t)$, where t stands for time, and a body force vector

$$\mathbf{F}=(F_\rho, F_\varphi, F_z)$$

such that

$$F_z=F_\varphi=0, \qquad F_\rho=F(\rho, t).$$

We assume that the displacement field

$$\mathbf{u}=(u_\rho, u_\varphi, u_z)$$

is axially symmetric and planar, i.e.,

$$u_z=u_\varphi=0, \qquad u_\rho=u(\rho, t). \tag{2.1}$$

Let $\sigma_{\rho\rho}$, $\sigma_{\varphi\varphi}$ . . . and $\epsilon_{\rho\rho}$, $\epsilon_{\varphi\varphi}$, . . . denote respectively the components of stress and strain in cylindrical coordinates. Then from (2.1) and the usual strain-displacement relations for these coordinates ([7], p. 183) it follows that

$$\epsilon_{\rho\rho}(\rho, t)=\frac{\partial u}{\partial\rho}(\rho, t), \qquad \epsilon_{\varphi\varphi}(\rho, t)=\frac{u(\rho, t)}{\rho}, \qquad \epsilon_{zz}=0$$

$$0=\epsilon_{\rho\phi}=\epsilon_{\varphi z}=\epsilon_{z\rho}$$

Let $G_1(t)$, $G_2(t)$ be respectively the relaxation functions in shear and isotropic compression corresponding to the base temperature T_0. We assume that the cylinder has been at rest at the reference temperature T_0 up to time $t=0$ and that the material with which we are dealing is thermorheologically simple.

In order to express concisely the plane strain stress-strain relations for a thermorheologically simple material and to avoid needlessly detailed equations in the sequel, we introduce the following operator notation. Given any differentiable function $G(t)$ defined on $[0,\infty)$ we associate with it the thermoviscoelastic operator V_G by defining[3]

$$V_G f(\rho, t)=G(0)f(\rho, t)+\int_0^t G^{(1)}\left(\int_{t'}^t \varphi(T(\rho,\tau))d\tau\right)\varphi(T(\rho, t'))f(\rho, t')dt'. \tag{2.3}$$

[2] In this paper, we have not attempted to determine the minimum sufficient smoothness of the given data. All given functions are assumed to be several times continuously differentiable.

[3] $G^{(1)}(u)=\dfrac{\partial G}{\partial u}$

Here, $T(\rho, t)$ is the temperature field introduced above, and $\varphi(T)$, the so-called shift factor, is a prescribed function which governs the change in material properties induced by a temperature field. The function φ has the properties

$$\varphi(T_0)=1,\ \varphi(T)>0,\ \frac{d\varphi}{dT}>0. \tag{2.4}$$

Notice that for $G(0)\neq 0$, V_G is a Volterra-type integral operator of the *second kind.* We shall assume that all the thermoviscoelastic operators which we encounter are of the second kind.

It is also convenient to define

$$M(t)=\tfrac{1}{3}(G_2(t)-G_1(t)) \tag{2.5}$$

$$G_1(0)=\gamma,\ M(0)=\mu \tag{2.6}$$

$$g(\rho,t,t')=G_1^{(1)}\left(\int_{t'}^{t}\varphi(T(\rho,\tau))d\tau\right)\varphi[T(\rho,t')] \tag{2.7}$$

$$m(\rho,\ t,\ t')=M^{(1)}\left(\int_{t'}^{t}\varphi(T(\rho,\ \tau))d\tau\right)\varphi[T(\rho,\ t')] \tag{2.8}$$

so that V_{G_1} and V_M take the form

$$V_{G_1}f(\rho,\ t)=\gamma f(\rho,\ t)+\int_0^t g(\rho,\ t,\ t')f(\rho,\ t')dt' \tag{2.9}$$

$$V_M f(\rho,\ t)=\mu f(\rho,\ t)+\int_0^t m(\rho,\ t,\ t')f(\rho,\ t')dt'. \tag{2.10}$$

Finally we introduce the function

$$\Phi(\rho,\ t)=\alpha_0 V_{G_2}\Theta(\rho,\ t). \tag{2.11}$$

The function Θ is defined by

$$\Theta(\rho,\ t)=\frac{1}{\alpha_0}\int_{T_0}^{T(\rho,\ t)}\alpha(T')dT',\ \alpha_0=\alpha(T_0)$$

and is known to thermoviscoelasticians as a "pseudotemperature." The prescribed material function $\alpha(T)$ is called the coefficient of linear thermal expansion.

The thermoviscoelastic stress-strain relations appear in many papers, for example Sternberg and Gurtin [2]. Written in terms of Cartesian coordinates and with the present operator notation they take the form

$$\sigma_{ij}(\rho,\ t)=V_{G_1}\epsilon_{ij}(\rho,\ t)+\delta_{ij}V_M\epsilon_{kk}(\rho,\ t)-\delta_{ij}\Phi(\rho,\ t). \tag{2.12}$$

In this equation, Latin subscripts take the range 1,2,3 and summation over repeated indices is implied. In the case of plane strain, Sternberg and Gurtin [2] apply Laplace transforms to solve for σ_{zz} in terms of the other stresses in the form

$$\sigma_{zz}=V_{\bar{\nu}}(\sigma_{\rho\rho}+\sigma_{\varphi\varphi})-\alpha_0 V_{\bar{E}}\Theta \tag{2.13}$$

where [4]

$$\bar{E}^*(\eta)=\frac{3G_1^*(\eta)G_2^*(\eta)}{G_1^*(\eta)+2G_2^*(\eta)}$$

$$\bar{\nu}^*(\eta)=\frac{G_2^*(\eta)-G_1^*(\eta)}{\eta(G_1^*(\eta)+2G_2^*(\eta)}.$$

[4] For f defined on $[0,\infty)$ $f^*(\eta)=\int_0^\infty e^{-\eta t}f(t)dt$.

327-016 O-69—3

From equations (2.12), which are invariant under a change of coordinate systems, and (2.2) we obtain the plane strain axisymmetric stress-strain relations

$$\begin{aligned}\sigma_{\rho\rho} &= V_{G_1}\left(\frac{\partial u}{\partial \rho}\right) + \mathrm{V}_M\left(\frac{\partial u}{\partial \rho}+\frac{u}{\rho}\right)-\Phi.\\ \sigma_{\varphi\varphi} &= V_{G_1}\left(\frac{u}{\rho}\right)+V_M\left(\frac{\partial u}{\partial \rho}+\frac{u}{\rho}\right)-\Phi\\ \sigma_{\rho\varphi} &= \sigma_{\varphi z}=\sigma_{z\rho}=0.\end{aligned} \tag{2.14}$$

Due to (2.13 and (2.14), the quasistatic stress equations of equilibrium ([7], page 184) reduce to the single equation

$$\frac{\partial}{\partial \rho}\sigma_{\rho\rho}+\frac{1}{\rho}(\sigma_{\rho\rho}-\sigma_{\varphi\varphi})+F=0. \tag{2.15}$$

If we substitute from (2.14) into (2.15) and apply the notation (2.9), (2.10) then the quasistatic displacement equation of motion takes the form

$$\begin{aligned}&\frac{\partial^2 u}{\partial \rho^2}(\rho, t)+\frac{\partial}{\partial \rho}\left(\frac{u(\rho, t)}{\rho}\right)\\ &\quad+\frac{1}{\gamma+\mu}\int_0^t\left\{(g(\rho, t, t')+m(\rho, t, t'))\left[\frac{\partial^2 u}{\partial \rho^2}(\rho, t')+\frac{\partial}{\partial \rho}\left(\frac{u(\rho, t')}{\rho}\right)\right]\right.\\ &\quad+\left(\frac{\partial g}{\partial \rho}(\rho, t, t')+\frac{\partial m}{\partial \rho}(\rho, t, t'))\frac{\partial u}{\partial \rho}(\rho, t')\right.\\ &\qquad\left.+\frac{\partial m}{\partial \rho}(\rho, t, t')\frac{u(\rho, t')}{\rho}\right\}dt'=H(\rho, t)\end{aligned} \tag{2.16}$$

where

$$H(\rho, t)=\frac{1}{\gamma+\mu}\left(\left(\frac{\partial \Phi}{\partial \rho}\right)(\rho, t)-F(\rho, t)\right). \tag{2.17}$$

Three types of boundary value problems are considered, namely,
(i) The displacement problem (first boundary value problem)

$$u(a, t)=u(b, t)=0,\ (t>0) \tag{2.18}$$

(ii) the traction problem

$$\sigma_{\rho\rho}(a, t)=\sigma_{\rho\rho}(b, t)=0, \text{ and} \tag{2.19}$$

(iii) the mixed problem

$$0=\sigma_{\rho\rho}(a, t)=u(b, t). \tag{2.20}$$

Our task then is to find a function $u(\rho, t)$ which satisfies (2.16) for (ρ, t) in $[a, b]\times[0, \infty)$ subject consecutively to each of the boundary conditions (2.18), (2.19), (2.20) where $\sigma_{\rho\rho}$ is defined in terms of u by (2.14). Stresses are then computed via (2.13), (2.14).

3. The solution

We define the differential operator E and the integrodifferential operator B as follows:

$$Ef(\rho, t)=\frac{\partial^2 f}{\partial \rho^2}(\rho, t)+\frac{\partial}{\partial \rho}\left(\frac{f(\rho, t)}{\rho}\right), \tag{3.1}$$

$$\begin{aligned} Bf(\rho, t)=\frac{1}{\gamma+\mu}\int_0^t \{&(g(\rho, t, t')+m(\rho, t, t'))Ef(\rho, t') \\ &+\left(\frac{\partial g}{\partial \rho}(\rho, t, t')+\frac{\partial m}{\partial \rho}(\rho, t, t')\right)\frac{\partial f}{\partial \rho}(\rho, t') \\ &+\frac{\partial m}{\partial \rho}(\rho, t, t')\frac{f(\rho, t')}{\rho}\Big\}\, dt'. \end{aligned} \tag{3.2}$$

Then (2.16) reduces to

$$Eu+Bu=H. \tag{3.3}$$

Our plan of attack is as follows: consider the model equation

$$Ef+Bf=P \tag{3.4}$$

for which we seek a solution $f(\rho, t)$ in $[a, b]\times[0, \infty)$ subject to the boundary conditions

$$M_a f(a, t)=M_b f(b, t)=0 \ (t>0). \tag{3.5}$$

Here M_a and M_b are some sort of linear integrodifferential boundary operators. Let E^{-1} be that operator which assigns to a given function $\psi(\rho, t)$ a unique function f which satisfies

$$Ef=\psi \tag{3.6}$$

in $[a, b[\times[0, \infty)$ and (3.5) for $t \geq 0$. Then clearly we obtain a solution f for (3.4), (3.5) by first finding a solution ψ for the equation

$$\psi+BE^{-1}\psi=P \tag{3.7}$$

and then setting

$$f=E^{-1}\psi. \tag{3.8}$$

The advantage to be gained by dealing with (3.7) rather than (3.4) lies in the fact that (3.7) is a sort of generalized integral equation and may be solved by the method of successive approximations. This decomposition of a viscoelastic problem into an elastic problem and an integral equation was introduced by Fichera in [8].

The formal solution of (3.7) is

$$\psi=(I+BE^{-1})^{-1}P$$

$$\psi=\sum_{n=0}^{\infty}(-1)^n(BE^{-1})^nP \tag{3.9}$$

so that

$$f=\sum_{n=0}^{\infty}(-1)^nE^{-1}(BE^{-1})^nP. \tag{3.10}$$

We shall now set forth sufficient conditions for the uniform convergence of the series (3.9), (3.10) in any rectangle $[a, b] \times [0, t]$ (t finite).

To this end, we define the norms [5]

$$\|h\|(t) = 1.u.b.\{|h(\rho, \tau)| : (\rho, \tau) \epsilon [a, b] \times [0, t]\} \tag{3.11}$$

$$\|h\|_1(t) = \left\|\frac{h}{\rho}\right\|(t) + \left\|\frac{\partial h}{\partial \rho}\right\|(t) \tag{3.12}$$

and postulate the existence of a nonnegative nondecreasing function $N(t)$ such that for any smooth function $\theta(\rho, t)$ and $t \geq 0$,

$$\|E^{-1}\theta\|_1(t) \leq N(t)\|\theta\|(t). \tag{3.13}$$

It is clear from (3.2) that for smooth g and m, B satisfies an inequality of the form

$$\|Bh\|(t) \leq Q(t) \int_0^t (\|Eh\|(t') + \|h\|_1(t'))dt'. \tag{3.14}$$

For example, one may take

$$Q(t) = \frac{1}{|\gamma + \mu|} \max \left\{1.u.b.\{|g_\rho(\rho, \tau, s) + m(\rho, \tau, s)| : (\rho, \tau, s) \epsilon [a, b] \times [0, t] \times [0, t]\},\right.$$

$$\left. 1.u.b. \left\{\left|\frac{\partial g}{\partial \rho}(\rho, \tau, s)\right| + \left|\frac{\partial m}{\partial \rho}(\rho, \tau, s)\right| : (\rho, \tau, s) \epsilon [a, b] \times [0, t] \times [0, t]\right\}\right\}.$$

The function Q so defined is clearly nonnegative and nondecreasing.

By replacing h by $E^{-1}\theta$ in (3.14) and applying (3.13) we obtain the estimate

$$\|BE^{-1}\theta\|(t) \leq Q(t)(1 + N(t)) \int_0^t \|\theta\|(t')dt'. \tag{3.15}$$

From (3.15), it follows by a standard iteration argument that

$$\|(BE^{-1})^n\theta\|(t) \leq \frac{(Q(t)(1 + N(t))t)^n}{n!} \|\theta\|(t). \tag{3.16}$$

Thus the series (3.9) converges uniformly in any rectangle $[a, b] \times [0, t]$. Since (3.13) implies that E^{-1} is continuous with respect to the uniform norm in such a rectangle, it follows that (3.10) also converges uniformly in $[a, b] \times [0, t]$ and that interchange of E^{-1} and the sum sign is permissible. Thus, sufficient conditions for the uniform convergence of (3.9) and (3.10) are the existence of the nondecreasing bounding functions $N(t)$ and $Q(t)$. The boundedness of Q follows if we assume that G_1, G_2, φ, and T are sufficiently smooth. The function $N(t)$ depends on the latter functions together with the boundary conditions.

Each of the boundary value problems (i), (ii), (iii) may be reduced to a problem of the form (3.4), (3.5). However, the operators M_a and M_b appearing in (3.5) will be different in each of the three cases. Thus, we must derive three different inverse operators $E_1^{-1}, E_2^{-1}, E_3^{-1}$ corresponding respectively to (i), (ii), and (iii) and find nonnegative, nondecreasing functions N_1, N_2, N_3 such that for any function φ and $t \geq 0$, E_j^{-1} and N_j satisfy (3.13). Then by the above theory, the solution of each of the three problems will have a uniformly convergent representation of the form (3.10).

[5] θ and h denote arbitrary smooth functions of ρ, t.

(i) *The Displacement Problem*

Let $E_1^{-1}\theta$ be that function h which satisfies

$$Eh(\rho, t)=\theta(\rho, t) \tag{3.17}$$

in $[a, b]\times[0, \infty)$

and

$$h(a, t)=h(b, t)=0 \tag{3.18}$$

for $t \geqslant 0$. By elementary techniques we obtain from (3.1), (3.17), and (3.18) the representation

$$E_1^{-1}\Theta(\rho, t)=\frac{1}{\rho}\int_a^{\rho}\Theta(y, t)\frac{(\rho^2-y^2)}{2}dy-\frac{(\rho^2-a^2)}{\rho(b^2-a^2)}\int_a^b\Theta(y, t)\frac{(b^2-y^2)}{2}dy. \tag{3.19}$$

It is obvious from (3.19) that a constant N_1 depending only on a and b exists such that

$$\|E_1^{-1}\Theta\|_1(t)\leqslant N_1\|\Theta\|(t). \tag{3.20}$$

Thus the displacement problem (2.18), (3.3) has the solution

$$u=\sum_{n=0}^{\infty}(-1)^nE_1^{-1}(BE_1^{-1})^nH, \tag{3.21}$$

where E_1^{-1} is given explicitly by (3.19).

(ii) *The Traction Problem*

For this problem, it is convenient to modify the operator notation (2.9), (2.10) as follows:

$$\begin{aligned}V_{G_1}^c[f(d, t')](t)&=\gamma f(d, t)+\int_0^t g(c, t, t')f(d, t')dt'\\ V_M^c[f(d, t')](t)&=\mu f(d, t)+\int_0^t m(c, t, t')f(d, t')dt'.\end{aligned} \tag{3.22}$$

Then by (3.3), (2.14), and (2.19) the traction problem takes the form

$$Eu+Bu=H \tag{3.3}$$

$$\Phi(a, t)=V_{G_1}^a\left[\frac{\partial u}{\partial\rho}(a, t')\right](t)+V_M^a\left[\frac{\partial u}{\partial\rho}(a, t')+\frac{u(a, t')}{a}\right](t) \tag{3.23}$$

$$\Phi(b, t)=V_{G_1}^b\left[\frac{\partial u}{\partial\rho}(b, t')\right](t)+V_M^b\left[\frac{\partial u}{\partial\rho}(b, t')+\frac{u(b, t')}{b}\right](t). \tag{3.24}$$

We seek a solution u for this problem in the form

$$u=v+w \tag{3.25}$$

where w is to be chosen in such a way that v satisfies a problem of the form (3.4), (3.5). Plugging (3.25) into (3.3) we obtain

$$Ev+Bv=H-Ew-Bw. \tag{3.26}$$

Also we demand of w that it satisfy the boundary conditions

$$\Phi(a,t)=V^a_{G_1}\left[\frac{\partial w}{\partial \rho}(a,t')\right](t)+V^a_M\left[\frac{\partial w}{\partial \rho}(a,t')+\frac{w(a,t')}{a}\right](t) \tag{3.27}$$

$$\Phi(b,t)=V^b_{G_1}\left[\frac{\partial w}{\partial \rho}(b,t')\right](t)+V^b_M\left[\frac{\partial w}{\partial \rho}(b,t')+\frac{w(b,t')}{b}\right](t). \tag{3.28}$$

Then u defined by (3.25) will satisfy (3.23), (3.24) provided v satisfies

$$0=V^a_{G_1}\left[\frac{\partial v}{\partial \rho}(a,t')\right](t)+V^a_M\left[\frac{\partial v}{\partial \rho}(a,t')+\frac{v(a,t')}{a}\right](t) \tag{3.29}$$

$$0=V^b_{G_1}\left[\frac{\partial v}{\partial \rho}(b,t')\right](t)+V^b_M\left[\frac{\partial v}{\partial \rho}(b,t')+\frac{v(b,t')}{b}\right](t). \tag{3.30}$$

The function w is by no means uniquely determined. We assume one of the form

$$w(\rho,t)=(\rho-a)^2\varphi_1(t)+(\rho-b)^2\varphi_2(t) \tag{3.31}$$

where the functions φ_1, φ_2 are to be determined by a substitution of (3.31) into (3.27) and (3.28). In this way we arrive at the equations

$$\Phi(a,t)=\left[2(\gamma+\mu)(a-b)+\mu\frac{(a-b)^2}{a}\right]\varphi_2(t)$$
$$+\int_0^t\left\{2(a-b)[g(a,t,t')+m(a,t,t')]+m(a,t,t')\frac{(a-b)^2}{a}\right\}\varphi_2(t')dt'$$

$$\Phi(b,t)=\left[2(\gamma+\mu)(b-a)+\mu\frac{(b-a)^2}{b}\right]\varphi_1(t)$$
$$+\int_0^t\left\{2(b-a)[g(b,t,t')+m(b,t,t')]+m(b,t,t')\frac{(b-a)^2}{b}\right\}\varphi_1(t')dt'.$$

Since these are, in general, Volterra-type integral equations of the second kind their solution presents no difficulties. We may thus regard w as known, so that the traction problem has been reduced to that of finding a function v which satisfies (3.26), (3.29) and (3.30). To this end, we define E_2^{-1} to be that operator which assigns to a given function $\theta(\rho,t)$ the function $h(\rho,t)$ which satisfies (3.17), (3.29), and (3.30). In order to derive E_2^{-1}, we write the general solution of (3.17) in the form

$$h(\rho,t)=\frac{1}{\rho}\int_a^\rho\theta(y,t)\left(\frac{\rho^2-y^2}{2}\right)dy+c_1(t)\rho+\frac{c_2(t)}{\rho}. \tag{3.32}$$

Then

$$\frac{\partial h}{\partial \rho}(\rho,t)=-\frac{1}{\rho^2}\int_a^\rho\theta(y,t)\left(\frac{\rho^2-y^2}{2}\right)dy+\int_a^\rho\theta(y,t)dy+c_1(t)-\frac{c_2(t)}{\rho^2}. \tag{3.33}$$

The functions $c_1(t)$, $c_2(t)$ are determined by a substitution of the expressions (3.32), (3.33) into (3.29), (3.30). They are governed by the equations

$$0=V^a_{G_1+2M}[c_1(t')](t)-\frac{1}{a^2}V^a_{G_1}[c_2(t')](t) \tag{3.34}$$

$$\Gamma(t)=V^b_{G_1+2M}[c_1(t')](t)-\frac{1}{b^2}V^b_{G_1}[c_2(t')](t) \tag{3.35}$$

where

$$\Gamma(t)=V^{b}_{G_1}\left[\frac{1}{b^2}\int_a^b\theta(y,t')\left(\frac{b^2-y^2}{2}\right)dy\right](t)-V^{b}_{G_1+M}\left[\int_a^b\theta(y,t')dy\right](t). \tag{3.36}$$

Except in singular cases, which we choose to disregard, all of the operators of the form V_G are Volterra-type integral operators of the second kind. Thus, they all have inverses, V_G^{-1}, which are operators of the same kind. Hence the system (3.34), (3.35) is readily solved, and we have

$$\begin{aligned}c_1(t)&=\frac{1}{a^2}V^{-1a}_{G_1+2M}V^{a}_{G_1}\left[\frac{1}{a^2}V^{-1a}_{G_1+2M}V^{a}_{G_1}-\frac{1}{b^2}V^{-1b}_{G_1+2M}V^{b}_{G_1}\right]^{-1}V^{-1b}_{G_1+2M}[\Gamma(t')](t)\\ c_2(t)&=\left[\frac{1}{a^2}V^{-1a}_{G_1+2M}V^{a}_{G_1}-\frac{1}{b^2}V^{-1b}_{G_1+2M}V^{b}_{G_1}\right]^{-1}V^{-1b}_{G_1+2M}[\Gamma(t')](t).\end{aligned} \tag{3.37}$$

An explicit representation for E_2^{-1} is obtained by combining (3.32), (3.37), and (3.36). It is clear from the form of these equations that there exists a nondecreasing function $N_2(t)$ such that

$$\|E_2^{-1}\theta\|_1(t)\leq N_2(t)\|\theta\|(t). \tag{3.38}$$

Consequently, we can write for the solution of the traction problem

$$u(\rho,t)=w(\rho,t)+\sum_{n=0}^{\infty}(-1)^nE_2^{-1}(BE_2^{-1})^n(H-Ew-Bw)(\rho,t). \tag{3.39}$$

(iii) *The Mixed Problem*

This problem is somewhat simpler than (ii). Thus, in the remainder of this paper we return to the earlier notation (2.9), (2.10).

We seek a function $u(\rho,t)$ which satisfies

$$Eu+Bu=H \tag{3.3}$$

$$\Phi(a,t)=V_{G_1}\left(\frac{\partial u}{\partial\rho}\right)(a,t)+V_M\left(\frac{\partial u}{\partial\rho}+\frac{u}{\rho}\right)(a,t) \tag{3.40}$$

$$0=u(b,t). \tag{3.41}$$

In order to deal with the inhomogeneity in (3.40), we again assume a solution of the form

$$u=v+w \tag{3.42}$$

where

$$Ev+Bv=H-Ew-Bw \tag{3.43}$$

$$0=V_{G_1}\left(\frac{\partial v}{\partial\rho}\right)(a,t)+V_M\left(\frac{\partial v}{\partial\rho}+\frac{v}{\rho}\right)(a,t) \tag{3.44}$$

$$0=v(b,t) \tag{3.45}$$

$$\Phi(a,t)=V_{G_1}\left(\frac{\partial w}{\partial\rho}\right)(a,t)+V_M\left(\frac{\partial w}{\partial\rho}+\frac{w}{\rho}\right)(a,t)$$

$$0=w(b,t).$$

An appropriate w is evidently given by

$$w(\rho, t) = (\rho - b)\varphi(t),$$

where φ satisfies

$$\Phi(a, t) = \left[\gamma + \mu\left(1 + \left(\frac{a-b}{a}\right)\right)\right]\varphi(t) + \int_0^t \left[g(a, t, t') + m(a, t, t')\left(1 + \left(\frac{a-b}{a}\right)\right)\right]\varphi(t')dt'.$$

Let E_3^{-1} be that operator which assigns to a given function $\theta(\rho, t)$ that function $h(\rho, t)$ which satisfies (3.17), (3.44), and (3.45).

We write the general solution of (3.17) in the form

$$h(\rho, t) = \frac{1}{\rho}\int_\rho^b \theta(y, t)\left(\frac{y^2 - \rho^2}{2}\right)dy + c_1(t)\rho + \frac{c_2(t)}{\rho}.$$

Condition (3.45) implies the further restriction

$$h(\rho, t) = \frac{1}{\rho}\int_\rho^b \theta(y, t)\left(\frac{y^2 - \rho^2}{2}\right)dy + c(t)\left(\rho - \frac{b^2}{\rho}\right). \tag{3.46}$$

Due to (3.44), $c(t)$ must satisfy the integral equation

$$\left[\left(1 + \frac{b^2}{a^2}\right)V_{G_1}^a + 2V_M^a\right][c(t')](t) = \frac{1}{a^2}V_{G_1}^a\left[\int_a^b \theta(y, t')\left(\frac{y^2 - a^2}{2}\right)dy\right](t)$$

$$+ V_{G_1 + M}^a\left[\int_a^b \theta(y, t')dy\right](t) \tag{3.47}$$

The existence of $N_3(t)$ is clear from (3.46), (3.47), so that the solution to the mixed problem takes the form

$$u(\rho, t) = w(\rho, t) + \sum_{n=0}^{\infty}(-1)^n E_3^{-1}(BE_3^{-1})^n(H - Ew - Bw)(\rho, t).$$

4. References

[1] Muki, R., and Sternberg, E., On transient thermal stresses in viscoelastic materials with temperature-dependent properties, J. Appl. Mech. **28**, No. 2, 193 (1961).

[2] Sternberg, E., and Gurtin, M. E., Further study of thermal stresses in viscoelastic materials with temperature-dependent properties, Proc. I.U.T.A.M., Symposium on second-order effects in elasticity, plasticity, and fluid dynamics, Haifa, 1962.

[3] Graham, G. A. C., On the solution of mixed boundary value problems, that involve time-dependent boundary conditions, for viscoelastic bodies with temperature-dependent properties, Grant No. AF-AFOSR-444-66, File No. PSR-47/7, North Carolina State University, April 1967.

[4] Edelstein, W. S., Existence of solutions to the displacement problem for quasistatic viscoelasticity, Arch. Rational Mech. Anal. **22**, No. 2, 121 (1966).

[5] Gurtin, M. E., and Sternberg, E., On the linear theory of viscoelasticity, Arch. Rational Mech. Anal. **11**, No. 4, 291 (1962).

[6] Sternberg, E., On the analysis of thermal stresses in viscoelastic solids, Proceedings of the 3rd Symposium on Naval Structural Mechanics, 349 (1963).

[7] Sokolnikoff, I. S., Mathematical theory of elasticity (McGraw-Hill Book Co., New York, 1956).

[8] Fichera, G., Lectures on linear elliptic differential systems and eigenvalue problems, Lecture Notes in Mathematics (Springer, New York, 1965).

[9] Valanis, K. C., and Lianis, G., Thermal stresses in a viscoelastic cylinder with temperature dependent properties, A.I.A.A. Journal, **2**, No. 9, 1942 (1964).

(Paper 73B1-284)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 73B, No. 1, January–March 1969

Relations Within Sequences of Congruential Pseudo-Random Numbers

Peter H. Verdier

Institute for Materials Research, National Bureau of Standards, Washington, D.C. 20234

(August 20, 1968)

Conditions are exhibited under which simple, approximate linear relations may be found between sets of successive choices made by congruential pseudo-random number generators. These relations imply that the distributions in n-dimensional space produced by such generators can be very highly nonuniform. The results are illustrated with several examples. Restrictions on the parameters of the generator to minimize difficulties of this sort are discussed.

Key Words: Congruential generators; multidimensional distributions; random number generators.

1. Introduction

Consider a congruential pseudo-random number generator of the form:

$$X_{j+1} \equiv MX_j + C \pmod{P}, \tag{1}$$

where all the quantities involved are nonnegative integers. Generators of this type are commonly used to provide sequences of pseudo-random numbers [1].[1] For implementation on digital computers, P is frequently chosen to be the word size of the computer and M is chosen as $2^a + N$ or $10^a + N$, where N is a small integer, for binary and decimal machines, respectively. In practice, such a generator is used to make a choice between some number B of alternatives. For this purpose, we form the quantity $[BX_j/P]$, where the square brackets denote integer part. It is the purpose of this paper to point out that relatively short sequences of such quantities may be very highly nonrandom, in the sense that if such a sequence is taken to represent a unit hypercube in a hypercube of side B, many of the possible unit hypercubes can never be selected. That is to say, knowledge of the last k choices produced by the generator gives us considerable information about what the next choice will be, for distressingly small k. Nonuniform distributions of triplets, corresponding to $k=2$, have been observed by MacLaren and Marsaglia [2], for several congruential generators. As a special case, the results in this paper partly explain their findings.

2. Results

Consider a generator of the form (1), with

$$M = R + N, \tag{2}$$

$$R^k \equiv r \pmod{P}, \tag{3}$$

[1] Figures in brackets indicate the literature references at the end of this paper.

where N, k, and r are integers small in magnitude. By successive application of eq (1), we obtain:

$$X_{j+m} \equiv M^m X_j + (M^m - 1)C/(M-1) \pmod{P}. \tag{4}$$

By using (3) and (4), we find:

$$\sum_{m=0}^{k} \binom{k}{m} (-N)^{k-m} X_m \equiv rX_0 + C\{R^k - (1-N)^k\}/(M-1) \pmod{P}, \tag{5}$$

where we have set $j = 0$ for convenience.

Equation (5) states that given the last k values of X, the next one is determined. This fact of itself is neither novel nor alarming; we already know from eq (1) that precise knowledge of just the last X tells us what the next one will be. However, we now show, with the aid of eq (5), that *approximate* knowledge of the last k values of X can enable us to predict the approximate value of the next. To show this, we suppose that the generator is to be used repeatedly to pick one of B alternatives, where B is a positive integer less than P. We first form from each X_j a new quantity b_j, defined by:

$$b_j = [BX_j/P], \tag{6}$$

so that $0 \leq b_j \leq B-1$. In effect, we have divided the range of values of X into B intervals, labeled by the values b_j. Then we can write:

$$BX_m/P = b_m + \epsilon_m, \tag{7}$$

where $0 \leq \epsilon_m \leq (P-1)/P < 1$. We now transform the congruence (5) into a congruence with modulus B. We first subtract the right-hand side of (5) from both sides, obtaining on the left-hand side an integer congruent to zero (mod P). Division of the left-hand side by P results by definition in an integer, and subsequent multiplication by B results, again by definition, in an integer congruent to zero (mod B). Rearranging terms to restore the transformed right-hand side, we obtain:

$$\sum_{m=0}^{k} \binom{k}{m} (-N)^{k-m} b_m \equiv rb_0 + \zeta - \eta \pmod{B}, \tag{8}$$

where we have set:

$$\zeta \equiv \frac{BC(R^k - (1-N)^k)}{P(M-1)} \pmod{B}, \tag{9.1}$$

$$\eta \equiv \sum_{m=0}^{k} \binom{k}{m} (-N)^{k-m} \epsilon_m - r\epsilon_0 \pmod{B}. \tag{9.2}$$

By considering the maximum possible sizes of the positive and negative terms in eq (9), using the fact that the ϵ's are all less than unity, we find from eq (8) that given the last k of the b's, the next one can only be one of S possibilities, where:

$$S = (1+|N|)^k + |(-N)^k - r| - |N|^k - 1, \text{ if } \zeta \text{ is an} \tag{10.1}$$

integer,

$$S = (1+|N|)^k + |(-N)^k - r| - |N|^k \text{ otherwise.} \tag{10.2}$$

(We ignore the trivial case $S=1$, obtained when the BX_m are all multiples of P.) If S is less than B, this means that there will be sequences of length $k+1$ which cannot occur at all. If we now choose:

$$R=d^a, \tag{11}$$

$$P=d^w, \tag{12}$$

where d is 2 or 10 for binary or decimal computers, respectively, we find that eq (3) is satisfied by $r=0$ and:

$$k=[(w-1)/a]+1, \tag{13.1}$$

the square brackets again denoting integer part. In the frequently employed special case of $C=0$ and M and all the X_j odd, eq (13.1) may be replaced by the stronger relation:

$$k=[(w-1-[\log_d 2])/a]+1. \tag{13.2}$$

3. Two Examples

In this section, we apply the results of the preceding section to two generators which have been discussed in the literature and found satisfactory by other tests. The first [3] is a multiplicative congruential generator with $M=2^{18}+3$, $C=0$, $P=2^{35}$, and the X_j all odd. From eq (13) we find $k=2$, so we expect a three-term recursion relation, corresponding to a set of parallel planes in three-space. From eq (8) we find that the equation obeyed is $b_{j+2}\equiv 6b_{j+1}-9b_j-\eta \pmod B$, and we can easily show from eq (9) that $-6<\eta<10$, leading to only 15 possibilities for b_{j+2}, given b_j and b_{j+1}. Thus, if this generator is used to select unit cubes at "random" in a cube of side greater than 15, some cubes will never be selected.

For the second example [4], we choose a mixed congruential generator with $M=2^{11}+1$, $C=1446722743$ octal, $P=2^{28}$. Then from eq (13), $k=3$, so we shall have a four-term recursion relation, given by eq (8). According to eq (10), if three successive b's are known, the next can only be one of 7 possibilities, if B is a multiple of 64, or 8 possibilities otherwise.

4. Discussion

In the preceding sections, we have shown that under certain circumstances, a generator of the form of eq (1), used with eq (6) to choose from among B possibilities, may have the property that knowing the last several choices made by the generator, we can predict that the next choice will be one of S possibilities, where S may be much smaller than B. This property constitutes a rather serious violation of the usual concept of a random choice. (Its disadvantages would be immediately obvious to, say, a casino operator who proposed to use such a generator in place of the conventional roulette wheel.) It should perhaps be noted that as one might expect from the form of eq (9), choice among the possibilities which *are* allowed is by no means made with equal frequency. Thus, to obtain the appearance of randomness, it is not sufficient merely to ensure that S is not less than B. Rather, one would expect that S needs to be considerably greater than B. Two examples may be found in the work of MacLaren and Marsaglia [2]. Among the generators for which they found poor triplet distributions were those with $M=2^{17}+3$, $C=0$, $P=2^{35}$, and $M=10^5+3$, $C=0$, $P=10^{10}$. For both generators, we find $k=2$, so that we expect triplets to be poorly distributed. From eq (10) we find $S=15$ in both cases. Since $B=10$ for the work in reference 2, no triplets are absolutely forbidden by eq (8). However, a chi-square test showed the distribution to be extremely nonuniform.

It is clear that in the cases of interest eq (10) will tend to be dominated by the first term on its right-hand side, i.e. $(1+|N|)^k$. The form of this term might suggest the use of a small multiplier, which from eq (3) will prohibit a small k. However, small multipliers give rise to other undesirable characteristics [5]. We need, therefore, to choose a nonsmall multiplier and a modulus such that for all possible decompositions of the multiplier according to eq (2) and (3), the value of S predicted by eq (10) is as large as possible (or, at any rate, considerably larger than some preselected number B of choices for which the generator is used). We do not see a way of going from eq (2), (3), and (10) to a formula for choosing M and P. However, these equations may be used to test a given trial M and P, in an obvious if somewhat laborious way.

As already remarked, generators of the forms implied by eqs (11) and (12) are especially common, because the implied division need not be performed explicitly. For generators of this form, eq (13) guarantees a recursion relation with $r=0$, for a value of k which will be small unless the multiplier is quite small. In this case, keeping S large amounts to requiring that the multiplier lie far from any multiple of all powers of d which are high enough to give rise to a small value of k. Again, it is not easy to see how this requirement can be turned into a prescription for choosing an optimum multiplier, but it is relatively easy to test trial multipliers.

If the modulus P is prime, eq (3) has no solutions with $r=0$. However, this does not make the choice of multiplier any easier. One can see from eq (10) that the effect of a small, nonzero value of r may be to decrease the value of S. Therefore, it is not clear that anything is gained by rejecting moduli of the form of eq (12).

Finally, we may note that whatever effect a nonzero constant C may have upon other statistical properties of a generator, it has virtually no effect upon the recursion properties discussed here. From eq (10), we see that a nonzero value of C can increase the number of possibilities by at most one, and that only for a suitable choice of B.

5. Conclusions

It seems clear that injudicious use of a congruential pseudo-random number generator can easily lead to highly nonuniform distributions in more than two dimensions, and that neither a nonzero additive constant nor a prime modulus will avoid this difficulty. Furthermore, routine testing of the multidimensional distributions produced by a generator may well not show up difficulties of the sort discussed here. Since B^k cells of storage are required to test a k-dimensional distribution, if k is greater than three, the storage capacities of present-day computers are likely to limit B to smaller values than those occurring in practice.

Note added in proof: A recent paper by G. Marsaglia [Proc. Nat. Acad. Sci. **61**, 25 (1968)] also deals with limitations on the randomness of n-tuples produced by congruential generators. The limitations established are weaker than those set forth here, but are independent of the choice of multiplier.

6. References

[1] For a review and bibliography, see Hull, T. E., and Dobell, A. R., Random number generators, SIAM Review **4**, 230–254 (1962).

[2] MacLaren, M. D., and Marsaglia, G., Uniform random number generator, J. Assoc. Comput. Mach. **12**, 83–89 (1965).

[3] Hull, T. E., and Dobell, A. R., Mixed congruential random number generators for binary machines, J. Assoc. Comput. Mach. **11**, 31–40 (1964).

[4] Peach, P., Bias in pseudo-random numbers, J. Am. Stat. Assoc. **56**, 610–618 (1961).

[5] Greenberger, M., An a priori determination of serial correlation in computer generated random numbers, Math. Comp **15**, 383–389 (1961); Math Comp. corrigenda **16**, 126 (1962).

(Paper 73B1–285)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 73B, No. 1, January–March 1969

On the Mean Dimensions of Restricted Random Walks

Peter H. Verdier and Edmund A. DiMarzio

Institute for Materials Research, National Bureau of Standards, Washington, D.C. 20234

(October 25, 1968)

A simple method is presented for obtaining mean dimensions of certain restricted random walks on lattices.

Key Words: Lattice; nonreversing walk; polymer dimension; random walk.

Random walks in three dimensions are frequently used as models of the possible configurations of certain forms of long-chain polymer molecules in solution [1].[1] In particular, walks are often used in which restrictions are imposed upon the angle between successive steps in the walk. Such restricted random walks are, of course, Markovian, and standard methods [2] are available for obtaining their mean dimensions. It is the purpose of this note to point out, however, that for a certain class of restricted random walks on a lattice, mean dimensions may be obtained by a simpler method which involves little more than an appropriate counting of configurations. The walks in question consist of steps of fixed length, with an equal-odds choice made at each step from a set of possible steps consisting of: a set of steps possessing a rotation-inversion symmetry axis, decreased or augmented by a step whose direction is determined by the direction of the preceding step. As a simple example, we consider the nonreversing walk on a simple cubic lattice of side unity in three dimensions. Here the choice at each step is made, with equal probability, from among five steps, four of them perpendicular to the last step, and the fifth continuing in the same direction. By way of example, we shall calculate the mean-square end-to-end length of this walk in two ways.

We start by writing

$$\langle l^2\rangle = \langle(\mathbf{r}_1 + \ldots + \mathbf{r}_N)^2\rangle = N + 2\sum_{j=1}^{N-1}\sum_{i=1}^{N-j}\langle \mathbf{r}_i \cdot \mathbf{r}_{i+j}\rangle, \tag{1}$$

where $\langle l^2\rangle$ is the mean-square end-to-end length of the walk, N is the number of steps, and the vector $\mathbf{r}_i$ represents the ith step. Let us consider the average dot product $\langle \mathbf{r}_i \cdot \mathbf{r}_{i+j}\rangle$. For the lattice in question, since there are five choices at each step, there are 5^j configurations over which to average. We can sort these out in two different ways, suggested by the identities:

$$5^j = (4+1)^j = 4^j + j\cdot 4^{j-1} + \ldots + 1 = \sum_{k=0}^{j}\binom{j}{k}4^{j-k}$$

$$= (6-1)^j = 6^j - j\cdot 6^{j-1} + \ldots + (-1)^j$$

$$= \sum_{k=0}^{j}\binom{j}{k}6^{j-k}(-1)^k.$$

In the first case, we regard the choices for each step as consisting of the set of four steps perpendicular to the last step, plus a step in the same direction. Then it is clear that in the first of the binomial expansions above, the first term is the number of j-step walks consisting entirely of right-angle steps, the second term is the number of walks with $j-1$ right-angle steps and one parallel step, and so forth. If we perform the sum of $\mathbf{r}_i \cdot \mathbf{r}_{i+j}$ over the 5^j configurations by summing over each

[1] Figures in brackets indicate the literature references at the end of this paper.

of these sets, we find that the sum over each set except the last is zero. To see this, we note that to every walk with a right-angle step in it there corresponds another walk produced by inverting the portion of the original walk that follows the right angle. Since $\mathbf{r}_{i+j}$ points in opposite directions for the two walks, their total contribution to $\langle \mathbf{r}_i \cdot \mathbf{r}_{i+j} \rangle$ is zero. (A similar argument may be made for lattices with a higher-order rotation-inversion axis, such as the triangular lattice in two dimensions.) Thus, the sum over the 5^j configurations reduces to the single term in the last set, which corresponds to the configuration with all steps parallel. For this configuration, $\mathbf{r}_i \cdot \mathbf{r}_{i+j}$ is unity, so we have $\langle \mathbf{r}_i \cdot \mathbf{r}_{i+j} \rangle = (1/5)^j$.

The same conclusion is also reached by regarding the choices at each set as consisting of the isotropic set of six choices, *less* the choice which constitutes reversal of the previous step. In this case, the first term in the second binomial expansion corresponds to completely unrestricted six-choice walks, the second term to walks with at least one reversal produced by forcing a reversal at one step of an otherwise unrestricted walk, and so on. As before, the contributions from all sets except the last are zero, and we are again left with the contribution from the single member of the last set, taken with coefficient $(-1)^j$. However, since this term corresponds to the configuration with a forced reversal at *every* step, the dot product will itself be $(-1)^j$, and we shall again have the result $\langle \mathbf{r}_i \cdot \mathbf{r}_{i+j} \rangle = (1/5)^j$. It is interesting to note that the sole contribution to the sum of the dot product over all configurations which do not reverse may be expressed in terms of the dot product value for the configuration which violates the restriction at every step.

Once $\langle \mathbf{r}_i \cdot \mathbf{r}_{i+j} \rangle$ has been obtained, evaluation of $\langle l^2 \rangle$ is straightforward. Since by definition of the random process $\langle \mathbf{r}_i \cdot \mathbf{r}_{i+j} \rangle$ is dependent only on the number j of intervening steps; we rewrite eq (1) in the form:

$$\langle l^2 \rangle = N + 2 \sum_{j=1}^{N-1} (N-j) \langle \mathbf{r}_i \cdot \mathbf{r}_{i+j} \rangle = N + 2 \sum_{j=1}^{N-1} (N-j)(1/5)^j. \tag{2}$$

Now, for $x \neq 1$, we have

$$\sum_{j=1}^{N-1} (N-j)x^j = Nx/(1-x) - x(1-x^N)/(1-x)^2$$

so that we can write

$$\langle l^2 \rangle = N[1 + 2x/(1-x)] - 2x(1-x^N)/(1-x)^2, \tag{3}$$

or with $x = 1/5$, $\langle l^2 \rangle = (3/2)N - (5/8)[1-(1/5)^N]$, in agreement with the result obtained for this walk by Domb and Fisher [3].

The foregoing argument may be applied to any isotropic nonreversing walk, in any number of dimensions. The result will be given by eq (3), with x equalling the reciprocal of the number of choices at each step. The argument can be applied equally well to what might be called a noncontinuing random walk, where continuation in the direction of the preceding step is barred, but the other possibilities, including reversal, are allowed. For such a walk, eq (3) again applies, but x is now the negative of the reciprocal of the number of allowed choices.

The same technique may be applied to walks where the choice is made from an isotropic set plus or minus *two* (or in principle, more) steps which depend upon the last step. As a trivial example, consider a random walk on a simple square or cubic lattice with both continuation and reversal disallowed. In this case, the last term in the appropriate binomial expansion is $(-2)^j$, and of this even number of terms, half have $\mathbf{r}_{i+j}$ pointing in the same direction as $\mathbf{r}_i$ and half have it pointing in the opposite direction. Hence all the nondiagonal dot products vanish, and we have immediately the result $\langle l^2 \rangle = N$. In the two-dimensional case, this result can, of course, be verified directly when N is even by redrawing the steps in pairs to form an unrestricted walk of $N/2$ steps, each of length $\sqrt{2}$.

References

[1] See, for example, Flory, P. J., Principles of Polymer Chemistry (Cornell University Press, Ithaca, N.Y., 1953).

[2] Flory, P. J., op. cit., ch. X.

[3] Domb, C., and Fisher, M. E., Proc. Camb. Phil. Soc. **54**, 48 (1958).

(Paper 73B1–286)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 73B, No. 1, January–March 1969

The Cube of Every Connected Graph is 1-Hamiltonian *

Gary Chartrand and S. F. Kapoor**

(November 15, 1968)

Let G be any connected graph on 4 or more points. The graph G^3 has as its point set that of G, and two distinct points u and v are adjacent in G^3 if and only if the distance between u and v in G is at most three. It is shown that not only is G^3 hamiltonian, but the removal of any point from G^3 still yields a hamiltonian graph.

Key Words: Cube of a graph; graph; hamiltonian.

Let G be a graph (finite, undirected, with no loops or multiple lines). A *walk* of G is a finite alternating sequence of points and lines of G, beginning and ending with a point and where each line is incident with the points immediately preceding and following it. A walk in which no point is repeated is called a *path*; the *length* of a path is the number of lines in it.

A graph G is *connected* if between every pair of distinct points there exists a path, and for such a graph, the *distance* between two points u and v is defined as the length of the shortest path if $u \neq v$ and zero if $u = v$.

A walk with at least three points in which the first and last points are the same but all other points are distinct is called a *cycle*. A cycle containing all points of a graph G is called a *hamiltonian cycle* of G, and G itself a *hamiltonian graph*.

Throughout the literature of graph theory there have been defined many graph-valued functions f on the class of graphs. In certain instances results have been obtained to show that if G is connected and has sufficiently many points, then the graph $f(G)$ (or its iterates $f^n(G)$) is a hamiltonian graph. Examples of such include the line-graph function $L(G)$ and the total graph function $T(G)$ (see [2, 1],[1] respectively).

The *line-graph* $L(G)$ of graph G is a graph whose point set can be put in one-to-one correspondence with the line set of G such that adjacency is preserved. The *total graph* $T(G)$ has its point set in one-to-one correspondence with the set of points and lines of G in such a way that two points of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent or incident.

Another example which always yields a hamiltonian graph is the cube function. In fact, if x is any line in a connected graph G with at least three points, then the cube of G has a hamiltonian cycle containing x. This follows from a result due to Karaganis [4] by which the cube of any connected graph G on $p(\geq 3)$ points turns out to be *hamiltonian-connected*, i.e., between any two points there exists a path containing all points of G. Now if x is any line joining points u and v in G, then the addition of x to the hamiltonian path between u and v in the cube of the graph produces a hamiltonian cycle of G containing x.

The *cube* G^3 of a connected graph G has as its point set that of G, and two distinct points u and v are adjacent in G^3 if and only if the distance between u and v in G is at most three. The purpose of this note is to prove that if G is a connected graph (with $p \geq 4$ points) then not only is G^3 hamiltonian, but the removal of any point from G^3 still yields a hamiltonian graph. Graphs enjoying this property have been referred to as 1-*hamiltonian* in [3].

THEOREM. *If* G *is a connected graph on* $p \geq 4$ *points, then* G^3 *is 1-hamiltonian.*

---

*An invited paper.

**Present address: Western Michigan University, Kalamazoo, Mich. 49001.

[1] Figures in brackets indicate the literature references at the end of this paper.

PROOF. That G^3 is hamiltonian is already known; indeed, there exists a hamiltonian cycle of G^3 which contains any specified line of G as we have already noted. We now show that the deletion of any point from G^3 also results in a hamiltonian graph. The proof is by induction on p, the result being true for $p=4$, since the cube of any connected graph on 4 points yields the complete graph, and the removal of any point from this leaves a cycle on three points. Clearly the theorem follows if the result is proved for a spanning tree T of G. Assume the statement to be true for all trees on n points, $4 \leq n < p$, and let T be any tree on p points. Consider the forest F obtained by deleting any point u from T. Denote the components of F by T_i, $1 \leq i \leq k$, where k is the degree of u in T. Furthermore, let T_i have p_i points so that $p = 1 + \sum_{i=1}^{k} p_i$, and let u_i be the point in T_i which is adjacent to u in T.

If $k=1$, then F is a tree. If $p_1 \geq 4$, then T_1^3 is 1-hamiltonian by hypothesis, while if $p_1=3$, T_1^3 is a triangle; in either case T_1^3 is hamiltonian.

Assume $k>1$. For all i such that $1 \leq i \leq k$ and $p_i \geq 3$, it follows from earlier remarks that a hamiltonian cycle C_i may be selected in T_1^3 to contain the line u_iv_i where v_i is adjacent to u_i in T_i. If $p_i=2$, T_i has only two points, namely u_i and v_i. We now construct a hamiltonian cycle C in T^3-u as follows.

For each C_i so defined for $p_i \geq 3$, remove the line u_iv_i to obtain a path P_i beginning at u_i and ending at v_i. If $p_i=2$, then T_i is itself a path P_i beginning at u_i and ending at v_i. For completeness, if $p_i=1$, we assign a second label v_i to the point u_i and speak of the trivial path P_i. Observe that for $r \neq s$, the distance from v_r to u_s is two if P_r is trivial, and is three otherwise. In either case the line v_ru_s is present in T^3-u. The desired cycle C is $P_1, v_1u_2, P_2, \ldots, P_j, v_ju_{j+1}, P_{j+1}, \ldots, P_k, v_ku_1$.

The previous result cannot be improved so that the removal of any two points from the cube of a connected graph with at least five points results in a hamiltonian graph. For example, if P is a path containing adjacent points u and v, neither an end-point, then $P^3-\{u, v\}$ is not hamiltonian.

References

[1] Behzad, M., Chartrand, G., Total graphs and traversability, Proc. Edinburgh Math. Soc. **15**, 117–120 (1966).

[2] Chartrand, G., The existence of complete cycles in repeated line-graphs, Bull. Amer. Math. Soc. **71**, 668–670 (1965).

[3] Chartrand, G., Kapoor, S. F., and Lick, D. R., n-Hamiltonian graphs, J. Combinatorial Theory, to appear.

[4] Karaganis, J. J., On the cube of a graph, Canad. Math. Bull. **11**, 295–296 (1968).

(Paper 73B1–287)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 73B, No. 1, January–March 1969

Minimum-Length Covering by Intersecting Intervals*

W. A. Horn

Institute for Basic Standards, National Bureau of Standards, Washington, D. C. 20234

(October 29, 1968)

This paper considers the problem: Given a sequence $\{I_i\}_1^n$ of intervals on the real axis, find a sequence $\{J_i\}_1^n$ of closed intervals which minimizes the sum-of-lengths $S=\Sigma_1^n|J_i|$ subject to $I_i \subseteq J_i$ and $J_i \cap J_{i+1} \neq \phi$ for all i. The paper gives a simple algorithm for determining the J_i and notes that linear programming can be applied to the more complicated problem where S is changed to $\Sigma_1^n \alpha_i|J_i|$, $\alpha_i > 0$.

Key Words: Covering problems; Manhattan metric.

1. Introduction

This paper deals with the following problem: Given a sequence $\{I_i\}_1^n$ of intervals on the real axis, find a sequence $\{J_i\}_1^n$ of closed intervals which minimizes the sum-of-lengths

$$S=\Sigma_1^n|J_i| \tag{1.1}$$

subject to the constraints

$$I_i \subseteq J_i \qquad (i=1, 2, \ldots, n), \tag{1.2}$$

$$J_i \cap J_{i+1} \neq \phi \qquad (i=1, 2, \ldots, n-1). \tag{1.3}$$

Interest in this problem arose while studying the determination of shortest length networks consisting only of vertical or horizontal lines connecting a set of points in the plane. If the points form "clusters," where each cluster lies on some vertical line, and if the clusters are far enough apart so that it is clear that only one horizontal line joins any two successive clusters, then the above problem arises as follows. For each cluster, it is clear that a vertical interval containing its highest (greatest ordinate) and lowest (smallest ordinate) point will be necessary and sufficient to cover the given cluster minimally. These vertical intervals correspond to the I_i, above. Since horizontal lines will be used to connect successive intervals, the intervals may need to be enlarged to some new intervals J_i, so that any two successive J_i have overlapping ordinates. Since the length of horizontal connecting lines is constant, a minimization of the above type results.

A simple solution algorithm for this problem is presented in section 2 and justified in section 3.

2. The Algorithm

Set $J_1=I_1$. Given $\{J_j\}_1^i$, J_{i+1} is found as follows:

(a) If $J_i \cap I_{i+1} \neq \phi$, set $J_{i+1}=I_{i+1}$.
(b) If $\max J_i < \min I_{i+1}$, set

$$J_{i+1}=[\max J_i, \max I_{i+1}].$$

*Supported in part by the Northeast Corridor Transportation Project, U.S. Department of Transportation. No official endorsement implied.

327-016 O-69—4

(c) If $\max I_{i+1} < \min J_i$, set

$$J_{i+1} = [\min I_{i+1}, \min J_i].$$

The process clearly yields a sequence $\{J_i\}_1^n$ which is *feasible*, i.e., satisfies (1.2) and (1.3). It remains to show that this sequence is also *optimal*, i.e., minimizes S over all feasible sequences.

3. Justification

The dependence of S on its arguments will be indicated by writing $S(\{J_i\}_1^n)$, while the minimum value of S for the problem with data $\{I_i\}_1^n$ will be denoted $S_{\min}(\{I_i\}_1^n)$.

The proof of optimality will be by induction on n. Clearly the algorithm yields an optimal result for $n=1$. Suppose this is true for $n=N$, and let $\{J_i\}_1^{N+1}$ be the result of applying the algorithm to data $\{I_i\}_1^{N+1}$. There are two cases to be considered.

CASE I: $I_1 \cap I_2 \neq \phi$. Then $J_2 = I_2$ holds, in addition to $J_1 = I_1$. Hence $\{J_i\}_2^{N+1}$ would result from applying the algorithm to $\{I_i\}_2^{N+1}$, so by the induction hypothesis

$$\Sigma_2^{N+1}|J_i| = S_{\min}(\{I_i\}_2^{N+1}).$$

Consider *any* $\{J_i^*\}_1^{N+1}$ feasible for $\{I_i\}_1^{N+1}$. J_1^* will cover I_1, and $\{J_i^*\}_2^{N+1}$ will be feasible for $\{I_i\}_2^{N+1}$, so that

$$S(\{J_i^*\}_1^{N+1}) = |J_1^*| + \Sigma_2^{N+1}|J_i^*|$$

$$\geqslant |I_1| + S_{\min}(\{I_i\}_2^{N+1}) = S(\{J_i\}_1^{N+1}),$$

proving that $\{J_i\}_1^{N+1}$ is optimal.

CASE II: $I_1 \cap I_2 = \phi$. Let $\{J_i^o\}^{N+1}$ be *any* sequence optimal for $\{I_i\}_1^{N+1}$. Define a sequence $\{J_i'\}_1^{N+1}$ as follows: $J_1' = I_1$, J_2' is the smallest interval covering J_2^o and meeting I_1, and $J_i' = J_i^o$ for $i > 2$. Clearly $\{J_i'\}_1^{N+1}$ is feasible for $\{I_i\}_1^{N+1}$, and it will be shown below that

$$|J_1'| + |J_2'| = |I_1| + |J_2'| = |J_1^o| + |J_2^o|, \tag{3.1}$$

implying $S(\{J_i'\}_1^{N+1}) = S(\{J_i^o\}_1^{N+1})$. Since $\{J_i^o\}_1^{N+1}$ is optimal, the same must be true of $\{J_i'\}_1^{N+1}$.

Now $\{J_i\}_2^{N+1}$ would result from applying the algorithm to the sequence $(J_2, \{I_i\}_3^{N+1})$, and so it is optimal for this sequence, by the induction hypothesis. But $\{J_i'\}_2^{N+1}$ is feasible with respect to this same sequence, since J_2' meets I_1 and covers I_2, while the algorithm generates J_2 as the smallest interval covering I_2 and meeting I_1. Hence

$$S(\{J_i\}_2^{N+1}) \leqslant S(\{J_i'\}_2^{N+1}),$$

and since $J_1 = J_1' = I_1$, it follows that

$$S(\{J_i\}_1^{N+1}) \leqslant S(\{J_i'\}_1^{N+1}).$$

Because $\{J_i'\}_1^{N+1}$ is optimal with respect to $\{I_i\}_1^{N+1}$, as noted above, the same must be true of $\{J_i\}_1^{N+1}$, as desired.

It only remains to prove (3.1). Since $\{J_i^o\}_1^{N+1}$ is optimal, J_1^o must be the smallest interval which covers I_1 and meets J_2^o. Recall that J_2' is the smallest interval which covers J_2^o and meets I_1.

Assume without loss of generality that $\max I_1 < \min I_2$. Then $\max I_1 < \max J_2^o$. If I_1 and J_2^o meet, then we must have $J_1^o = I_1$ and $J_2' = J_2^o$ so that (3.1) holds. Therefore, suppose $\max I_1 < \min J_2^o$.

Then

$$J_1^0 = [\min I_1, \min J_2^0], \quad J_2' = [\max I_1, \max J_2^0],$$

and so

$$\begin{aligned} |I_1| + |J_2'| &= (\max I_1 - \min I_1) + (\max J_2^0 - \max I_1) \\ &= \max J_2^0 - \min I_1 \\ &= (\max J_2^0 - \min J_2^0) + (\min J_2^0 - \min I_1) \\ &= |J_2^0| + |J_1^0|. \end{aligned}$$

This completes the justification of the algorithm.

4. Differently Weighted Intervals

A problem similar to the above but slightly more complicated is the following. Minimize

$$S = \Sigma_1^n \alpha_i |J_i| \tag{4.1}$$

subject to constraints (1.2) and (1.3) of section 1, where all $\alpha_i > 0$.

Although no algorithm as simple as the above has been found to solve this case, it is worth noting that the minimization can be performed by a linear program, as follows.

Let $I_i = [a_i, b_i]$ and $J_i = [A_i, B_i]$. Then the problem is minimized when

$$S = \Sigma_1^n \alpha_i (B_i - A_i)$$

is minimized, subject to

$$A_i \leqslant a_i, B_i \geqslant b_i (i = 1, 2, \ldots, n), \tag{4.2}$$

and

$$A_i \leqslant B_{i+1}, B_i \geqslant A_{i+1} (i = 1, 2, \ldots, n-1), \tag{4.3}$$

where (4.2) is equivalent to (1.2) and (4.3) is equivalent to (1.3). It is clear that this formulation results in a linear program in the variables A_i and B_i.

(Paper 73B1–288)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 73B, No. 1, January–March 1969

Publications of the National Bureau of Standards*

Selected Abstracts

Mather, J. N., **Invariance of the homology of a lattice,** *Proc. Am. Math. Soc.* **17,** *No. 5, 1120–1124 (Oct. 1966).*
Rota, Kan, Peterson and Whitehead have developed a homology theory for finite lattices which *a priori* depends upon the choice of a cross-cut of the lattice. It is shown here that the different cross-cuts of a lattice lead to simplicial complexes which have the same homotopy type and are therefore homology equivalent.

Key Words: Homology; lattice; combinatorics; cross-cut; simplicial complex; homotopy.

Newman, M., **A bound for the number of conjugacy classes in a group,** *J. London Math. Soc.* **43,** *108–110 (1968).*
It is shown that if the group G is of order h and has k conjugacy classes, then $k \geq \log \log h/\log 4$.

Key Words: Conjugacy classes; bounds; groups; matrix representations.

Olver, F. W. J., **An extension of Millers's algorithm,** *Aplikace Mat.* **13,** *174–176 (1968).*
A new algorithm is described for computing the solution of any second-order linear difference equation which is applicable when simple recurrence procedures cannot be used because of instability.

Key Words: Chebyshev series; difference equations; error analysis; Miller algorithm; recurrence methods; special functions.

Other NBS Publications

J. Res. NBS 73A (Phys. and Chem.), No. 1 (Jan.-Feb. 1969), $1.00.
Specific heats of oxygen at coexistence. R. D. Goodwin and L. A. Weber.
Specific heats C_v of fluid oxygen from the triple point to 300 K at pressures to 350 atmospheres. R. D. Goodwin and L. A. Weber.
Thermodynamic properties of fluid oxygen at temperatures to 250 K and pressures to 350 atmospheres on isochores at 1.3 to 3.0 times critical density. R. D. Goodwin.
Two new standards for the pH scale. B. R. Staples and R. G. Bates.
Calculation of diffusion coefficients in ternary systems from diaphragm cell experiments. P. R. Patel, E. C. Moreno, and T. M. Gregory.
Calculated line strength for the transition array $(3d^3 + 3d^24s) - 3d^24p$ in Ti II. H. Mendlowitz.
Morphological stability of a cylinder. S. R. Coriell and S. C. Hardy.
Distorted tetrahedra in strontium copper åkermanite. J. Ito and H. S. Peiser.
Tritium-labeled compounds XII. Note on the synthesis of D-glucose-2-*t* and D-mannose-2-*t*. H. S. Isbell, H. L. Frush, C. W. R. Wade, and A. J. Fatiadi.
A survey of blemishes on processed microfilm. C. S. McCamy, S. R. Wiley, and J. A. Speckman.

J. Res. NBS 72C (Engr. and Instr.), No. 4 (Oct.-Dec. 1968), 75 cents.
Theoretical and experimental study on longitudinal impact of tapered rods. L. R. Hettche.
Analytical studies of probe conduction errors in ground temperature measurements. B. A. Peavy.
Nonlinear constrained optimization by a nonrandom complex method. R. A. Mitchell and K. L. Kaplan.
Carter, G. C., Bennett, L. H., Cuthill, J. R., Kahan, D. J., The NBS alloy data center: Function, bibliographic system, related data centers, and reference books, Tech. Note 464 (Aug. 1968), $1.25.
Colson, J. P., Clark, E. S., Tables of solutions to Bragg's equation for copper, cobalt, iron and chromium K^- radiation and small diffraction angles, Tech. Note 461 (Aug. 1968), 50 cents.
Cord, M. S., Lojko, M. S., Petersen, J. D., Microwave spectral tables, Volume V. Spectral line listing, Mono. 70, Volume V (June 1968), $4.75.
Davis, M. M., Acid-base behavior in aprotic organic solvents, Mono. 105 (Aug. 1968), $2.25.
deMaine, P. A. D., Input/output packages for the systems 360 assembly language processor, Tech. Note 433 (Sept. 1968), 25 cents.
Eisele, J. W., Tech. Std. Coordinator, Fabrics for book covers, PS9–68 (Apr. 15, 1968), 10 cents.
Fraction, G. F., Walker, J. C., Tauber, S. J., Connection tables from Wiswesser chemical structure notations—a partial algorithm, Tech. Note 432 (Sept. 1968), 65 cents.
Goldman, D. T., Ed., Neutron cross sections and technology, Proc. Conf., Washington, D. C., Mar. 4–7, 1968, Spec. Publ. 299 (Sept. 1968), $10.50 set of 2 volumes.
Haar, L., Thermodynamic properties of ammonia as an ideal gas, NSRDS–NBS 19 (Aug. 1968), 20 cents.
Heinrich, K. F. J., Ed., Quantitative electron probe microanalysis, Proc. Seminar, National Bureau of Standards, Gaithersburg, Md., June 12–13, 1967, Spec. Publ. 298 (Oct. 1968), $3.00.
Johnston, H. S., Gas phase reaction kinetics of neutral oxygen species, NSRDS–NBS 20 (Sept. 1968), 45 cents.
Lovell, W. S., Thiel, L. M., Interferometric measurements of the complex dielectric constant of liquids, Tech. Note 369 (Aug. 1968), 50 cents.
Marsden, C. P., Tabulation of published data on Soviet electron devices through October 1967, Tech. Note 441 (July 1968), 55 cents. (Supersedes Tech. Note 265).
Menis, O., Ed., Analytical Coordination Chemistry Section: Summary of activities July 1967 to June 1968, Tech. Note 454 (July 1968), 55 cents.
Moore, C. E., Bibliography on the analyses of optical spectra. Section 1. ^{1}H–^{23}V, Spec. Publ. 306 (Sept. 1968), $1.00.
Muhlenhaupt, R. C., Strobridge, T. R., An analysis of the Brayton cycle as a cryogenic refrigerator, Tech. Note 366 (Aug. 1968), $1.25.
NBS frequency and time broadcast services radio stations WWV, WWVH, WWVB, and WWVL, Spec. Publ. 236, 1968 edition (1968), 15 cents. (Supersedes Misc. Publ. 236, 1967 edition.).
Scribner, B. F., Ed., Spectrochemical Analysis Section: Summary of Activities July 1967 to June 1968, Tech. Note 452 (Sept. 1968), $1.00.
Swanson, H. E., McMurdie, H. F., Morris, M. C., Evans, E. H., Standard X-ray diffraction powder patterns, Mono. 25, Sect. 6 (June 1968), 60 cents.
Swindells, J. F., Ed., Precision measurement and calibration. Selected NBS papers on temperature, Spec. Publ. 300, Vol. 2 (Aug. 1968), $4.75.
Tate, D. R., Acceleration due to gravity at the National Bureau of Standards, Mono. 107 (June 1968), 25 cents.
Wachtman, J. B., Jr., Franklin, A. D., Ed., Mass transport in oxides. Proceedings of a symposium held at Gaithersburg, Maryland, October 22-25, 1967, Spec. Publ. 296 (Aug. 1968), $3.00.

Watstein, D., Johnson, P. V., Experimental determination of eccentricity of floor loads applied to a bearing wall, Bldg. Sci. Series 14 (June 1968), 15 cents.

Wegstein, J. H., Rafferty, J. F., Pencak, W. J., Matching fingerprints by computer, Tech. Note 466 (July 1968), 20 cents.

Abrams, M. D., A comparative sampling of the systems for producing computer-drawn flow charts, Proc. 1968 ACM National Conf., Las Vegas, Nevada, Aug. 27-29, 1968, pp. 743–750 (Brandon/Systems Press, Inc., Princeton, N.J., 1968).

Ambs, W. J., Field-emission studies of the oxidation of clean nickel surfaces, Proc. 2nd Intern. Congress on Metallic Corrosion, New York, N.Y., Mar. 11-15, 1963, pp. 815–821 (Natl. Assoc. Corrosion Engr., Houston, Texas, 1966).

Armstrong, G. T., Calorimetry, Encyclopaedic Dictionary of Physics, 2nd Supplementary Volume **2**, 1–8 (Pergamon Press Inc., New York, N.Y., Oct. 1967).

Ausloos, P., Lias, S. G., Primary modes of decomposition of superexcited *n*-alkane molecules (Proc. Intern. Conf. Photochemistry, Munchen, Germany, Sept. 6-9, 1967), Ber. deut. Gunsenges. **72**, No. 2, 187–195 (1968).

Ausloos, P., Rebbert, R. E., Lias, S. G., Structure and reactivity of propylions in gas-phase radiolysis, J. Am. Chem. Soc. **90**, No. 18, 5031–5033 (Aug. 28, 1968).

Bates, A. A., Responsibility in building research, Proc. Third Australian Building Research Cong., Melbourne, Australia, Aug. 1967, pp. 27–34 (1968).

Bates, R. G., Measurement of pH, Handbook of Biochemistry, Ed. H. A. Sober, pp. J100–J194 (Chemical Rubber Company, Cleveland, Ohio, 1967).

Beaty, E. C., Patterson, P. L., Mobilities and reaction rates in neon ions in neon, Phys. Rev. **170**, No. 1, 116–121 (June 5, 1968).

Bechtoldt, C. J., Ogburn, F., Smit, J., Structure and morphology of electrodeposited molybdenum dendrites, J. Electrochem. Soc. **115**, No. 8, 814–816 (Aug. 1968).

Bender, P. L., Alley, C. O., Currie, D. G., Faller, J. E., Satellite geodesy using laser range measurements only, J. Geophys. Res. **73**, No. 16, 5353–5357 (Aug. 15, 1968).

Bender, P. L., Branscomb, L. M., Credit for gravity apparatus, Phys. Today Letter **20**, No. 10, 17 (Oct. 1967).

Bennett, H. S., F center in ionic crystals: semicontinuum polaron models, Phys. Rev. **169**, No. 3, 729–745 (May 15, 1968).

Bennett, J. A., The initiation of fatigue microcracks under sequential bending and torsional loads, Trans. ASM **61**, No. 2, 210–218 (June 1968).

Bevan, A. W., Jr., Daniel, A. C., Mahler, R. J., Indication of the effect of a magnon gap in the ^{19}F relaxation rate in antiferromagnetic CoF_2, Physics Letters **25A**, No. 7, 555–556 (Oct. 9, 1967).

Branscomb, L. M., The misinformation explosion: Is the literature worth reviewing? Sci. Res. **3**, No. 11, 49–52 (May 1968).

Broberg, J., Leiss, J., Schrack, R. A., Wyckoff, J. M., On-line handling system for the NBS linac, IEEE Trans. Nucl. Sci. **NS-11**, No. 3, 331–337 (June 1964).

Bullis, W. M., Brewer, F. H., Kolstad, C. D., Swartzendruber, L. J., Temperature coefficient of resistivity of silicon and germanium near room temperature, Solid-State Electron. **11**, No. 7, 639–646 (July 1968).

Burke, P. G., Cooper, J. W., Ormonde, S., Electron-impact excitation of $n=2$ state in He, Phys. Rev. Letters **17**, No. 7, 345–348 (Aug. 15, 1966).

Bussey, H. E., Instrumentation and measurement, 1966 Digest of Literature on Dielectrics **30**, Publ. 1496, 1–21 (National Academy of Sciences-National Research Council, Washington, D.C., 1966).

Chang, T., Yang, G. C., Temperature dependence of paramagnetic relaxation of Mo^{5+} in TiO_2, J. Chem. Phys. **48**, No. 6, 2546–2549 (Mar. 15, 1968).

Chappell, S. E., Humphreys, J. C., Motz, J. W., Berger, M. J., Seltzer, S. M., Response of silicon transmission detectors to monoenergetic electrons, IEEE Trans. Nuclear Sci. **NS-15**, No. 3, 359 (June 1968).

Codling, K., Madden, R. P., Structure in the $L_{II,III}$ absorption of aluminum and its oxides, Phys. Rev. **167**, No. 3, 587–591 (Mar. 1968).

Cooper, M. J., Electric field effect on critical opalescence of dipolar binary mixture, J. Chem. Phys. **48**, No. 9, 4272–4277 (May 1, 1968).

Corliss, E. L. R., A mechanistic model for the limits of auditory preception, Proc. Symp. on Models for the Perception of Speech and Visual Form, Boston, Mass., Nov. 11–14, 1964, pp. 339–343 (Massachusetts Institute of Technology, Boston, Mass., 1967).

Daniel, A. C., Bevan, A. W., Jr., Mahler, R. J., Nuclear spin-lattice relaxation in the antiferromagnets $CoCl_2 \cdot 2H_2O$ and $FeCl_2 \cdot 2H_2O$, J. Appl. Phys. **39**, No. 2, Part I, 496–497 (Feb. 1, 1968).

Davis, J. C., Faison, T. K., Achenbach, P. R., Errors in temperature measurement in moving air under isothermal conditions using thermocouples, thermistors and thermometers, ASHRAE Trans. **73**, Part I, VII 1.1 (1967).

Dean, J. W., Richards, R. J., Hydrostatic pressure effects in carbon and germanium thermometers (Proc. 1967 Cryogenic Engineering Conf., Stanford University, Stanford, Calif., Aug. 21–23, 1967), Book, Advances in Cryogenic Engineering **13**, 505–508 (Plenum Press Inc., New York, N.Y., 1968).

Drechsel, D., Maximon, L. C., Potential model calculation for coplanar and non-coplanar proton-proton bremsstrahlung, Physics Letters **26B**, No. 8, 477–479 (Mar. 1968).

Drechsel, D., Seaborn, J. B., Greiner, W., Collective correlations in spherical nuclei and the structure of giant resonances. Phys. Rev. **162**, No. 4, 983–991 (Oct. 1967).

Dunn, G. H., Photodissociation of H_2^+ and D_2^+: theory, Phys. Rev. **172**, No. 1–7 (Aug. 5, 1968).

Durst, R. A., Ross, J. W., Jr., Electrochemical generation of fluoride ion by solid-state transference, Anal. Chem. **40**, No. 8, 1343–1344 (July 1968).

Evenson, K. M., The optical detection of stimulated emission in CN at 20-CM wavelength, Appl. Phys. Letters **12**, No. 8, 253–254 (Apr. 15, 1968).

Forbes, E. J., A graphic method of economic feasibility analysis (Proc. Inst. Electronics and Automation in Publishing, Washington, D.C., 1965), Book, Automation and Electronics in Publishing, Ed., L. H. Hattery and G. P. Bush, **3**, Part IV, Chapt. 13, 129–140 (Spartan Books, Washington, D.C., 1965).

Forman, R. A., Cardona, M., Exciton electroreflectance in II–VI compounds, Proc. Intern. Conf. II–VI Semiconducting Compounds, Brown University, Providence, Rhode Island, Sept. 6–8, 1967, pp. 100–111 (W. A. Benjamin, Inc., New York, N.Y., 1967).

Fowler, H. A., Experiments in Electron scattering. 1, Proc. Symp. Quantitative Aspects of Electron Microscopy, Walter Reed Institute of Pathology, Washington, D.C., 1965; Lab. Investigations **14**, No. 6, Part 2, 9–16 (1965).

Franklin, A. D., NBS dedicates new laboratory, J. Am. Ceramic Soc. **45**, No. 10, 841–842 (1966).

Freeman, D. H., Aiyar, A. S., Studies of ion exchange materials: homogeneous fractional sulfonation of copolymers of styrene and divinylbenzene, Anal. Chem. **39**, No. 10, 1141–1144 (Aug. 1967).

Freeman, D. H., Paulson, R. A., Chemical microstandards from ion exchange resin, Nature **218**, 563–564 (May 11, 1968).

Gallagher, A., Rubidium and cesium excitation transfer in nearly adiabatic collisions with inert gases, Phys. Rev. **172**, No. 1, 88–96 (Aug. 5, 1968).

Gammon, W. H., Remotes and displays, Proc. A Pioneer Presentation of a National Symposium on the Impact of Automation and Documentation, Denver, Colo., Apr. 27–29, 1967, p. 119 (1968).

Gammon, W. H., Security and privacy problems brought on by automation, Proc. A Pioneer Presentation of a National Symposium on the Impact of Automation and Documentation, Denver, Colo., Apr. 27–29, 1967, pp. 74–75 (1968).

Gammon, W. H., Software as it relates to documentation, Proc. A Pioneer Presentation of a National Symposium on the Impact of Automation and Documentation, Denver, Colo., Apr. 27–29, 1967, pp. 14–22 (1968).

Gammon, W. H., What educators can expect from the center for Computer Sciences and Technology, Proc. Gen. Conf. Midwestern States Educational Information Project, Des Moines, Iowa, June 20, 1967, P. L. 89–10, Title V, Sect. 505 pp. 30–45 (Department of Public Instruction, State of Iowa, Des Moines, Iowa, 1967).

Gates, D. M., Keegan, H. J., Weidner, V. R., Spectral reflectance and planetary reconnaissance (Proc. Third Goddard Memorial Symp. Scientific Experiments—Manned Orbital Flights, Washington, D.C., Mar. 18–19, 1965), Book, Scientific Experiments for Manned Orbital Flights **4**, 71–86 (Western Periodicals Co., North Hollywood, Calif., 1965).

Glasgow, A. R., Jr., High-temperature valve for corrosive gases, Anal. Chem. **38,** 1104 (July 1966).

Goldman, D. T., Reactor physicists emphasize need for neutron data, Phys. Today **21,** No. 8, 85–89 (Aug. 21, 1968).

Gould, H. A., Guernsey, R. L., Williams, R. H., Comment on the kinetic equation of a dilute gas with repulsive and attractive interactions, J. Chem. Phys. Letter **47,** No. 2, 872–873 (July 15, 1967).

Greenberg, L., Kleinian groups, Proc. Am. Math. Soc. **17,** No. 5, 1195–1198 (Oct. 1966).

Gugler, A. L., A study of the adherence of porcelain enamel to aluminum, Proc. Porcelain Enamel Institute Forum **29,** 202–208 (Nov. 1967).

Haber, S., A combination of Monte Carlo and classical methods for evaluating multiple integrals, Bull. Am. Math. Soc. **74,** No. 4, 683–686 (July 1968).

Hall, M. B., Little, W. E., A directional coupler with a readily calculable coupling ratio, IEEE Trans. Microwave Theory Tech. **MTT-15,** No. 11, 598–603 (Nov. 1967).

Harris, W. P., Low frequency dielectric behavior (Proc. Seventh Electrical Insulation Conf., Chicago, Ill., Oct. 16–19, 1967), IEEE Publ. **32C79-42,** 123–125 (1967).

Heinrich, K. F. J., Electron probe microanalysis by specimen current measurement (Proc. Fourth Intern. Symp. X-ray Optics and Microanalysis, Orsay, France, Sept. 7–10, 1965), Book, X-ray Optics and Microanalysis, Eds. R. Castaing, P. Deschamps, and J. Philibert, pp. 159–167 (Hermann, Paris, France, 1966).

Heinrich, K. F. J., Scanning electron probe microanalysis, Am. Soc. Testing Mater. Spec. Tech. Publ. **430,** 315–342 (1968).

Heinrich, K. F. J., Vieth, D., Yakowitz, H., Correction for nonlinearity of proportional counter systems in electron probe x-ray microanalysis, Advan. X-ray Anal. **9,** 208–220 (1966).

Herber, R. H., Spijkerman, J. J., Narrow-line source for ^{119}Sn Mössbauer spectroscopy, J. Chem. Phys. **43,** No. 11, 4057–4059 (Dec. 1, 1965).

Herron, J. T., Huie, R. E., Arrhenius parameters for the reactions of atomic nitrogen with some olefins and acetylenes, J. Phys. Chem. **72,** No. 7, 2538–2540 (July 1968).

Herron, J. T., Huie, R. E., On the reaction of atomic nitrogen with carbon dioxide, J. Phys. Chem. **72,** No. 6, 2235–2236 (June 1968).

Hertelendy, P., An approximate theory governing symmetric motions of elastic rods of rectangular or square cross section, J. Appl. Mech. Paper No. 68–**APM–10,** pp. 333–341 (June 1968).

Hetzer, H. B., Robinson, R. A., Bates, R. G., Dissociation constants of piperazinium ion and related thermodynamic quantities from 0 to 50°, J. Phys. Chem. **72,** 2081–2086 (1968).

Hiza, M. J., Heck, C. K., Kidnay, A. J., Liquid-vapor and solid-vapor equilibrium in the system hydrogen-ethane (Proc. 1967 Cryogenic Engineering Conf., Stanford University, Stanford, Calif., Aug. 21–23, 1967), Book, Advances in Cryogenic Engineering **13,** 343–356 (Plenum Press Inc., New York, N.Y., 1968).

Hobbs, A. M., Remarks on network simplification, J. Operations Res. **15,** No. 3, 548–551 (May–June 1967).

Hughes, E. E., Dorko, W. D., Direct mass spectrometric determination of atmospheric carbon dioxide, Anal. Chem. **40,** No. 6, 866–869 (May 1968).

Hummer, D. G., Radiative transfer processes in planetary nebulae (Proc. Intern. Astronomical Union, Symp. No. 34, Tatranska Lomnica, Czechoslovakia, Sept. 4–8, 1967), Book, Planetary Nebulae, Ed. D. E. Osterbrock and C. R. O'Dell, pp. 166–184 (D. Reidel Publ. Co., Dordrecht, Amsterdam, The Netherlands, 1968).

Hummer, D. G., Mihalas, D., Line formation with non-coherent electron scattering in O and B stars, Astrophys. J. **150,** No. 1, L57–L59 (Oct. 1967).

Ito, J., Silicate apatities and oxyapatites, Am. Mineralogist **53,** 890–907 (May–June 1968).

Ito, J., Strontium rare-earth hydro-germanate and garnets, Mater. Res. Bull. **3,** 495–500 (1968).

Jackson, J. L., Coriell, S. R., Transport coefficients of composite materials, J. Appl. Phys. **39,** No. 5, 2349–2354 (Apr. 1968).

Johannesen, R. B., Candela, G. A., Tsang, T., Jahn-Teller distortion: magnetic studies of vanadium tetrachloride, J. Chem. Phys. **48,** No. 12, 5544–5549 (June 15, 1968).

Kerns, D. M., Beatty, R. W., Book, Basic theory of waveguide junctions and introductory microwave network analysis, 150 pgs. (Pergamon Press Inc., London, England, 1967).

Kidnay, A. J., Hiza, M. J., Dickson, P. F., The adsorption isotherms of methane, nitrogen, hydrogen and their mixtures on charcoal at 76° K (Proc. 1967 Cryogenic Engineering Conf., Stanford University, Stanford, Calif., Aug. 21–23, 1967), Book, Advances in Cryogenic Engineering 13, 397–408 (Plenum Press Inc., New York, N.Y., 1968).

Klemens, P. G., Jackson, J. L., Diffusion of electrons on the Fermi surface. II. Longitudinal magnetoresistance of Ib metals, Physica **31,** 1421–1427 (1965).

Klein, R., The deposition of hydrogen beams on tungsten, Surface Sci. **11,** 227–241 (1968).

Koch, H. W., Caswell, R. S., Muehlhause, C. O., Neutron measurement at the National Bureau of Standards, Nuclear News **10,** No. 11, 48–53 (Nov. 1967).

Kopec, C. S., Measurement of gears, Handbook of Industrial Metrology, Chapt. 15, pp. 424–450 (Prentice-Hall, Inc., Englewood Cliffs, N.J., 1967).

Kokoszka, G. F., Allen, H. C., Jr., Gordon, G., Electron paramagnetic resonance spectra of two zinc-doped and nickel-doped copper chloride pyridine-*N*-oxide complexes, J. Chem. Phys. **46,** No. 8, 3020–3024 (Apr. 15, 1967).

Kokoszka, G. F., Allen, H. C., Jr., Gordon, G., Magnetic and optical spectra of copper monochloracetate 2.5 hydrate, J. Chem. Phys. **47,** No. 1, 10–13 (July 1, 1967).

Krauss, M., Use of approximate Hartree-Fock calculations to estimate ionization potentials, Book, Advances in Mass Spectrometry **4,** 575–578 (Elsevier Publ. Co., Amsterdam, The Netherlands, Aug. 14, 1967).

Kruger, J., The spectrum of corrosion research of the Corrosion Section of the National Bureau of Standards (Proc. 12th Annual Appalachian Underground Corrosion Short Course, West Virginia University, Morgantown, West Virginia, Oct. 1967), West Virginia University Tech. Bull. **86,** 11–13 (1967).

Landgrebe, A. R., Pella, P. A., McClendon, L. T., DeVoe, J. R., Purdy, W. C., The application of substoichiometric radioisotopic dilution principles to controlled potential coulometry and solvent extraction, Anal. Chim. Acta **39,** 151–159 (1967).

Larsen, S. Y., Quantum-mechanical pair-correlation function of hard spheres, J. Chem. Phys. **48,** No. 4, 1701–1708 (Feb. 15, 1968).

Lederer, P. S., Performance testing, pressure transducers, Instr. Control System **40,** No. 9, 93–99 (Sept. 1967).

Leiss, J. E., Electron linear accelerators as sources of high intensity neutron bursts, Proc. Seminar on Intense Neutron Sources, Santa Fe, New Mexico, Sept. 19–23, 1966, Report No. CONF–660925, pp. 605–616 (1966).

Leiss, J. E., The NBS linac, Proc. Linear Accelerator Conf., Los Alamos, New Mexico, Oct. 3–7, 1966, Report No. LA–3609, p. 20 (1966).

Lenzi, M., Okabe, H., Photodissociation of NOCl and NO_2 in the vacuum ultraviolet (Proc. Intern. Conf. Photochemistry, Munich, Germany, Sept. 6–9, 1967), Ber. deut. Bunsenges. Phys. Chem. **72,** No. 2, 169–173 (1968).

Leslie, R. T., Kuehner, E. C., Ebulliometry, Treatise on Analytical Chemistry **8,** Part D–5, Chapt. 89, 5085–5109 (1968).

Lightbody, J. W., Jr., Penner, S., A 12-channel semiconductor counter system for the NBS electron scattering spectrometer, IEEE Trans. Nucl. Sci. **NS–15,** 419–425 (June 1968).

Linzer, M., Forman, R. A., NMR studies of single crystal ND_4Cl, J. Chem. Phys. **46,** No. 12, 4690–4693 (June 15, 1968).

Litovitz, I., Macedo, P. B., Ultrasonic relaxation, viscosity, and free volume in molten glasses (Proc. Intern. Conf. Physics of Non-Crystalline Solids, Delft, The Netherlands, July 6–10, 1964), Book, Physics of Non-Crystalline Solids, Ed. J. A. Prins, p. 220 (North Holland Publ. Co., Amsterdam, The Netherlands, 1965).

Loftus, T. P., Services and facilities for the calibration of soft x-ray detection instruments at the National Bureau of Standards (Proc. Conf. Detection and Measurement of X-radiation from Color Television Receivers, Washington, D.C., Mar. 28–29, 1968), Book, Conference on Detection and Measurement of X-radiation from Color Television Receivers, pp. 129–134 (U.S. Dept. Health, Education and Welfare, Washington, D.C., Mar. 1968).

Lonie, M., A report on body sizing for boys and students, Boys' and Young Men's Apparel Buyers' Assoc. J., p. 8 (1967).

Lutz, G. J., DeSoete, D. A., Determination of carbon in sodium by photon activation analysis, Anal. Chem. **40,** No. 4, 820–822 (Apr. 1968).

McCulloh, K. E., Franck-Condon calculations for double ionization of molecular hydrogen, J. Chem. Phys. **48,** No. 4 2090–2093 Mar. 1, 1968).

McCulloh, K. E., Rosenstock, H. M., Experimental test of the Franck-Condon principle: Double ionization of molecular hydrogen, J. Chem. Phys. **48,** No. 4, 2084–2089 (Mar. 1, 1968).

McMurdie, H. F., Progress in X-ray diffraction data compilations, Chapter in Fifty Years of Progress in Metallorgraphic Techniques, Am. Soc. Testing Mater. Spec. Tech. Publ. **430,** 192–200 (1968).

Macek, J. H., Application of the Fock expansion to doubly excited states of the helium atom, Phys. Rev. **160,** No. 1, 170–174 (Aug. 5, 1967).

Maeda, K., Young, J., Propagation of the pressure waves produced by auroras, J. Geomagnetism Geoelectricity **18,** No. 2, 275–299 (1966).

Mahaffey, C. T., Basic construction standards and a technical information center recommended at a Pan American seminar, ASA Mag. Std. **36,** No. 12, 375–377 (Dec. 1965).

Maienthal, E. J., Taylor, J. K., Improvement of polarographic precision by a comparative technique, Mikrochim. Acta **5,** 939–945 (1967).

Maki, A. G., Interpretation of the CS_2 laser transitions, Appl. Phys. Letters **11,** No. 6, 204–205 (Sept. 1967).

Mann, D. E., Acquista, N., Linevsky, M. J., An electronic band spectrum of yttrium monofluoride (Proc. 4th Rare Earth Research Conf., Phoenix, Ariz., Apr. 22–25, 1964), Book, Proceedings of the Fourth Conference on Rare Earth Research, pp. 363–372 (Gordon and Breach, New York, N.Y., 1965).

Mann, D. E., White, D., Seshadri, K. S., Dever, D. F., Linevsky, M. J., The infrared spectra, structures and thermodynamics of gaseous LiO, Li_2O, and Li_2O_2, J. Chem. Phys. **39,** 2463–2473 (1963).

Marezio, M., Remeika, J. P., Zocchi, M., Cation distribution in the garnet $Eu_3Fe_{1.94}Ga_{3.06}O_{12}$, J. Chem. Phys. **48,** No. 3, 1094–1096 (Feb. 1968).

Margoshes, M., Selection of wavelengths for atomic absorption spectrometry, Anal. Chem. **39,** No. 10, 1093–1096 (Aug. 1967).

Marton, L., Photoelectricity, Encyclopedia of Physics, Ed. R. M. Besancon, pp. 512–514 (1966).

Maryott, A. A., Birnbaum, G., Line shape and collision effects in the microwave wing of far-infrared rotational lines, J. Chem. Phys. **47,** No. 9, 3200–3205 (Nov. 1967).

Melmed, A. J., Field emission microscopy, Book, Advances in Materials Research, Volume I. Experimental Methods, Ed. H. Herman, pp. 103–143 (Interscience Publ. Inc., New York, N.Y., 1967).

Melmed, A. J., Whiskers, thin films, and applications (real and imagined) to mass spectrometry and biological molecule imaging, Book, Field-Ion Microscopy, Ed. J. J. Hren and S. Ranganathan, pp. 183–212 (Plenum Press Inc., New York, N.Y., 1968).

Melmed, A. J., Layer, H. P., Kruger, J., Ellipsometry, low energy electron diffraction and field emission microscopy combined, Surface Sci. **9,** 476–483 (1968).

Miller, K. J., Krauss, M., Inapplicability of the born exchange scattering amplitude for the He ($1^1S \rightarrow 2^3S$) transition, J. Chem. Phys. **48,** No. 6, 2611–2614 (Mar. 15, 1968).

Moiseiwitsch, B., Smith, S. J., Electron impact excitation of atoms, Rev. Mod. Phys. **40,** No. 2, 238–353 (Apr. 1968).

Moore, G. A., Automatic scanning and computer processes for the quantitative analysis of micrographs and equivalent subjects (Proc. Symp. Automatic Photo Interpretation, Washington, D.C., May 31–June 2, 1967), Book, Pictorial Pattern Recognition, Ed. G. C. Cheng, R. S. Ledley, D. K. Pollock, and A. Rosenfeld, pp. 275–326 (Thompson Book Co., Washington, D.C., 1968).

Mopsik, F. I., Molecular and ionic interactions, 1966 Digest of Literature on Dielectrics **30,** Chapt. 3, 62–90 (National Academy of Sciences-National Research Council, Washington, D.C., 1966).

Mountain, R. D., Relaxation times for the volume viscosity of simple fluids, J. Chem. Phys. **48,** No. 5, 2189–2190 (Mar. 1, 1968).

Mountain, R. D., Zwanzig, R., Anomalous transport properties of a van der Waals gas, J. Chem. Phys. **48,** No. 4, 1451–1458 (Feb. 15, 1968).

Mulroy, W. J., Kusuda, T., Phillips, C. W., Survey on digital computer programs for heating and cooling load calculations, SICCAP Bull. **2,** No. 2, 4–8 (May 1968).

Odabasi, H., Spectrum of doubly ionized lanthanum (La III), J. Opt. Soc. Am. **57,** 1459 (1967).

Parker, R. L., Report on the 1966 Boston International Conference on Crystal Growth, Proc. 1966 Intern. Union Crystallography, Moscow, U.S.S.R., July 12–21, 1966, 7, 341–354 (Academy of Sciences, Moscow, U.S.S.R., 1968).

Patty, O. L., Little, W. E., Zanboorie, M. H., Waveguide connector measurements with a millimeter wave reflectometer, IEEE Trans. Microwave Theory Tech. **MTT-16,** No. 2, 121–123 (Feb. 1968).

Penner, S., Handling high power electron beams, IEEE Trans. Nucl. Sci. **NS-14,** 908 (June 1967).

Penner, S., Handling high-energy high-power electron beams, Record IEEE 9th Annual Symp. Electron, Ion and Laser Beam Technology, Berkeley, Calif., May 9–11, 1967, No. F-79, 50–61 (1967).

Penney, R. W., Phillips, C. W., Refrigeration requirements for truck bodies—effects of door usage, USDA Tech. Bull. No. 1375 (Superintendent of Documents, Government Printing Office, Washington, D.C. 20402, Nov. 1967, 10 cents).

Persson, K-B., Johnson, E. G., Jr., Uhlenbrock, D. A., A theory for cyclotron harmonic radiation from plasmas, Phys. Fluids **1,** No. 3, 619–628 (Mar. 1968).

Peterson, R. L., Ising chain as the basic cluster in effective-field theories of magnetism, Physics Letters **27A,** 177 (June 10, 1968).

Peterson, R. L., Suppressed spin-lattice relaxation, Phys. Rev. Letters **20,** No. 24, 1346–1348 (June 10, 1968).

Rhodes, I., The importance of the glossary storage in machine translation, Book, Machine Translation, Ed. A. D. Booth, pp. 431–449 (North-Holland Publ. Co., Amsterdam, The Netherlands, 1967).

Rhodes, I., The mighty man-computer team, Blanch Anniversary Volume, pp. 277–284 (Aerospace Research Laboratories, Office of Aerospace Research, U.S. Air Force, Arlington, Va., Feb. 1967).

Robbins, C. R., Synthetic $CaTiSiO_5$ and its germanium analogue ($CaTiGeO_5$), Mater. Res. Bull. **3,** 693–698 (1968).

Robbins, R. F., Reed, R. P., Tensile properties of neoprene below the glass transition temperature (Proc. 1967 Cryogenic Engineering Conf., Stanford University, Stanford, Calif., Aug. 21–23, 1967), Book, Advances in Cryogenic Engineering **13,** 252–258 (Plenum Press Inc., New York, N.Y., 1968).

Roder, H. M., The thermal conductivity of helium gas at moderate pressures, Proc. Thermodynamik Symp., Heidelberg, Germany, Sept. 1967, Sponsored by the International Union of Pure and Applied Chemistry, Ed. K. L. Schafer, Paper No. VI-3, pp. 1–8 (1968).

Ronn, A. M., Laser induced infrared fluorescence, J. Chem. Phys. **48,** No. 1, 511–513 (Jan. 1968).

Rubin, R. J., Momentum autocorrelation function of a heavy particle in a finite crystal, J. Am. Chem. Soc. **90,** 3061–3063 (1968).

Rybicki, G. B., Hummer, D. G., Spectral line formation in variable-property media: the Riccati method, Astrophys. J. **150,** No. 2, 607–635 (Nov. 1967).

Sadowski, W. L., On some aspects of the eigenfunction expansion of the solution of the nonlinear Vlasov equation (Proc. Computer Simulation of Plasma and Many-Body Problems, Williamsburg, Va., Apr. 19–21, 1967), Natl. Aeronaut. Space Agency Spec. Publ. **153,** 443–440 (1967).

Scribner, B. F., Advances in excitation sources for spectrochemical analysis, Pure Appl. Chem. **10,** 579–594 (July 21, 1967).

Schafft, H. A., Avoiding second breakdown, Proc. XIII Intern. Scientific Congress on Electronics, Rome, Italy, June 1966, pp. 119–130 (Rassegna Internazionale Elettronica Nucleare Teleradiocinematografica, Rome, Italy, 1966).

Schindler, P. W., Heterogeneous equilibria involving oxides, hydroxides, carbonates and hydroxide carbonates (Proc. Symp. Division of Water, Air, and Waste Chemistry, 151st Meeting of the American Chemical Society, Pittsburgh, Pa., Mar. 1966), Book, Equilibrium Concepts in Natural Water Systems, Advances in Chemistry Series **67,** 196–221 (American Chemical Society, New York, N.Y., 1967).

Schwerdtfeger, W. J., Discussion of the paper "Method of estimating corrosion of highway culverts by means of polarization curves," by R. I. Lindberg, Highway Research Record No. 204, Metal Corrosion, Publ. 1535, p. 8 (National Research Council-National Academy of Sciences, Washington, D.C., 1967).

Sengers, J. V., Triple collision contribution to the transport coefficients of gases (Lectures presented at the Theoretical Physics Institute, University of Colorado, Boulder, Colo., Summer 1966), Book, Lectures in Theoretical Physics, Vol. **IXC.** Kinetic Theory, Ed. W. E. Brittin, pp. 335–374 (Gordon and Breach, New York, N.Y., 1967).

Shapiro, J. T., Madden, R. P., On the optical constants of polystyrene in the vacuum ultraviolet, J. Opt. Soc. Am. **58,** No. 6, 771–775 (June 1968).

Shideler, R. W., Interfacing a teletypewriter to nuclear instruments used for computer-coupled activation analysis, Proc. 1965 Intern. Conf. Modern Trends in Activation Analysis, College Station, Texas, Apr. 1965, pp. 236–242 (1965).

Shimizu, A., Calculations of the penetration of gamma rays through slabs by the method of invariant imbedding, Nucl. Sci. Eng. **32,** 184–194 (May 1968).

Shimizu, A., Calculation of the penetration of gamma rays through two-layer slabs, Nucl. Sci. Eng. **32,** 385–391 (June 1968).

Shuler, K. E., Reaction cross sections, rate coefficients and non-equilibrium kinetics (Lecture, NATO Summer School on Theoretical Chemistry, Constance, Germany, Summer 1965), Book, Chemische Elementarprozesse, Ed. H. Hartmann, pp. 1–22 (Springer-Verlag, Berlin, Germany, 1968).

Silverman, S., Reflections on bioengineering (Proc. Symp. Engineering Significance of the Biological Sciences, Pittsburgh, Pa., Jan. 25–27, 1967), Book, Bioengineering—An Engineering View, pp. 103–106 (San Francisco Press Inc., San Francisco, Calif., June 1968).

Simmon, J. H., Macedo, P. B., High temperature shear ultrasonic interferometer using sensitive phase-lock detection system, J. Acoust. Soc. Am. **43,** No. 6, 12–95–1301 (June 1968).

Smith, G. W., Becker, D. A., Preparation of an NBS biological standard reference material for trace elemental analysis, Proc. Nuclear Activation Techniques in the Life Sciences, Amsterdam, The Netherlands, May 8–12, 1967, pp. 197–207 (International Atomic Energy Agency, Vienna, Austria, 1967).

Spokas, O. E., The NBS linac master control and personnel protection system, IEEE Trans. Nucl. Sci. **NS–14,** 1016 (June 1967).

Stern, J., Aspect systems, Encyclopedia of Library and Information Science 1, 3200 (1967).

Sukle, D. J., Wells, J. S., EPR sample orientation servo, Rev. Sci. Instr. Notes **39,** No. 4, 604–605 (Apr. 1968).

Suzuki, G., Quantitative methods for decision making, Book, Management: Concepts and Practices, Ed. F. R. Brown, Chapt. VIII, pp. 155–193 (Industrial College of the Armed Forces, Washington, D.C., 1967).

Tighe, N. J., Hyman, A., Transmission electron microscopy of alumina ceramics (Proc. Intern. Symp. Anisotropy in Single-Crystal Refractory Compounds, Dayton, Ohio, June 13–15, 1967), Book, Anisotropy in Single-Crystal Refractory Compounds, Ed. F. W. Vahldiek and S. A. Mersol, **2,** 121–136 (Plenum Press Inc., New York, N.Y., 1968).

Tisone, G. C., Branscomb, L. M., Detachment of electrons from H^- and O^- negative ions by electron impact, Phys. Rev. **170,** No. 1, 169–183 (une 5, 1968).

Truesdell, C., The non-linear field theories of mechanics, Encyclopedia of Physicas III, Part 3, 602 pages (Springer-Verlag, Berlin, Germany, 1965).

Tschuikow-Roux, E., Photolysis of propene at 1470Å, J. Phys. Chem. Note **71,** 2355 (1967).

Ugiansky, G. M., Ellinger, G. A., Corrosion of Monel 400 in high CO_2 well water, Corrosion **24,** No. 5, 134–136 (May 1968).

Ugiansky, G. M., Skolnick, L. P., Kruger, J., Stiefel, S. W., Rate-controlling step in stress corrosion cracking, Nature **218,** No. 5147, 1156–1157 (June 22, 1968).

Utech, H. P., Early, J. G., On the presence of thermal convection in the kinetics experiments of Rigney and Blakely, Acta. Met. **15,** No. 7, 1238–1239 (1967).

Utech, H. P., Parker, R. L., Early, J. G., Coriell, S. R., The cam equation for a thermal-wave generator, J. Appl. Phys. **38,** No. 9, 3799–3800 (Aug. 1967).

VanderHart, D. L., Gutowsky, H. S., Farrar, T. C., Dipole-dipole interactions of a spin ½ nucleus with a quadrupole-coupled nucleus, J. Am. Chem. Soc. **89.** No. 19, 5056–5057 (Sept. 13, 1967).

Vriens, L., Kuyatt, C. E., Mielczarek, S. R., Tests of born approximations: differential and total cross sections for elastic scattering of 100- to 400-eV electrons by helium, Phys. Rev. **170,** No. 1, 163–169 (June 1968).

Wachtman, J. B., Jr., Brower, W. S., Farabaugh, E. N., Elastic constants of single crystal calcium molybdate ($CaMoO_4$), J. Am. Ceram. Soc. **51,** No. 6, 341–344 (June 1968).

Waclawski, B. J., Hughey, L. R., Madden, R. P., Effect of oxygen adsorption on the photoelectron yield from tungsten in the vacuum ultraviolet, Appl. Phys. Letters **10,** No. 11, 305–307 (June 1968).

Wahsweiler, H. G., Greiner, W., Danos, M., Continuum nuclear structure of O^{16} in the eigenchannel reaction theory, Phys. Rev. **170,** No. 4, 893–906 (June 20, 1968).

Wang, F. E., DeSavage, B. F., Buehler, W. J., The irreversible critical range in the TiNi transition, J. Appl. Phys. **39,** No. 5, 2166–2175 (Apr. 1968).

Wassink, H. W., Cyclotron harmonic emission along the field from a plasma, Phys. Fluids **11,** No. 3, 629–637 (Mar. 1968).

Watson, T. H., Flynn, D. R., Thermal conductivity and electrical resistivity of beryllium copper foil, Trans. Met. Soc. **AIME 242,** 876–880 (May 1968).

Watson, T. W., Flynn, D. R., Thermal conductivity of four heat-resistant alloys, Trans. Met. Soc. **AIME 242,** 844–846 (May 1968).

Weisman, H. M., Information and the discipline of communication sciences (Proc. American Documentation Institute Annual Meeting, New York, N.Y., Oct. 22–27, 1967), Book, Levels of Interaction Between Man and Information **4,** 8–12 (Thompson Book Co., Washington, D.C., 1968).

Weissberg, A., Data handling systems, Proc. Second Conf. Technical Information Center Administration, St. David's, Pa., June 14–17, 1965, TICA 2, Ed. A. W. Elias, Chapt. 7, pp. 75–89 (Spartan Books, Washington, D.C. 1965).

Weitzel, D. H., Sindt, C. F., Daney, D. E., Hydrogen slush density reference system (Proc. 1967 Cryogenic Engineering Conf., Stanford University, Stanford, Calif., Aug. 21–23, 1967), Book, Advances in Cryogenic Engineering **13,** 523–533 (Plenum Press Inc., New York, N.Y., 1968).

Wiederhorn, S. M., The influence of water vapor on crack propagation in soda-lime glass, J. Am. Ceram. Soc. **50,** No. 8, 407 (1967).

Wiese, W. L., Transition probabilities for prominent Ar I lines (Proc. 8th Intern. Conf. Phenomena in Ionized Gases, Vienna, Austria, Aug. 27–Sept. 2, 1967), Book, 8th International Conference on Phenomena in Ionized Gases, p. 447 (Springer Verlag, Vienna, Austria, Sept. 1967).

Winogradoff, N. H., Neill, A. H., Band tailing effects and the temperature dependence of radiative recombination in compensated, epitaxial GaAs laser junctions, IEEE J. Quantum Elect. **QE–4,** No. 4, 111–113 (Apr. 1968).

Wyckoff, J. M., On-line transformations of high resolution spectra, IEEE Trans. Nucl. Sci. **NS–14,** 634 (Feb. 1967).

Wyckoff, J. M., Radioactivity produced by a linac, IEEE Trans. Nucl. Sci. **NS–14,** 990 (June 1967).

Yakowitz, H., Vieth, D. L., Heinrich, F. J., Michaelis, R. E., Homogeneity characterization of NBS spectrometric standards. II: Cartridge brass and low-alloy steel (Proc. 14th Annual Conf. Applications of X-ray Analysis, University of Denver, Denver Research Inst., Denver, Colo., Aug. 25–27, 1965), Book, Advances in X-ray Analysis, Ed. G. R. Mallett, M. Fay, and W. M. Mueller, **9,** 289–303 (Plenum Press Inc., New York, N.Y., 1966).

Yee, K. W., Deslattes, R. D., Transistorized current stabilizer for X-ray tubes with directly heated cathodes, Rev. Sci. Instr. **38,** No. 5, 637–638 (May 1967).

Yolken, H. T., Kruger, J., Thermal regeneration of oxide covered iron {100} and {110} surfaces, J. Electrochem. Soc. **114,** No. 8, 796–799 (Aug. 1967).

Yolken, H. T., Kruger, J., Calvert, J. P., Hydrogen in passive films on Fe, Corrosion Sci. **8,** 103–108 (1968).

Younglove, B. A., Polarizability, dielectric constant, pressure, and density of solid parahydrogen on the melting line, J. Chem. Phys. **48,** No. 9, 4181–4186 (May 1, 1968).

Zwanzig, R., Lauritzen, J. I., Jr., Exact calculation of the partition function for a model of two-dimensional polymer crystallization by chain folding, J. Chem. Phys. **48,** No. 8, 3351–3360 (Apr. 15, 1968).

**Publications for which a price is indicated are available by purchase from the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 20402 (foreign postage, one-fourth additional). The NBS non-periodical series are also available from the Clearinghouse for Federal Scientific and Technical Information, Springfield, Va. 22151. Reprints from outside journals and the NBS Journal of Research may often be obtained directly from the authors.*

U.S. GOVERNMENT PRINTING OFFICE : 1969 OL—327-016

UNITED STATES
GOVERNMENT PRINTING OFFICE
DIVISION OF PUBLIC DOCUMENTS
WASHINGTON, D.C. 20402

OFFICIAL BUSINESS

POSTAGE AND FEES PAID
U.S. GOVERNMENT PRINTING OFFICE

JOURNAL OF RESEARCH of the National Bureau of Standards Vol. 73B, No. 1 January–March 1969
B. Mathematical Sciences

Contents

	Page
A table of integrals of the error functions. Edward W. Ng and Murray Geller.	1
Fourier coefficients of Mathieu functions in stable regions. Henning Früchting.	21
A note on the *T*-transformation of Lubkin. W. D. Clark, H. L. Gray, and J. E. Adams.	25
The cylinder problem in thermoviscoelasticity. Warren S. Edelstein.	31
Relations within sequences of congruential pseudo-random numbers. Peter H. Verdier.	41
On the mean dimensions of restricted random walks. Peter H. Verdier and Edmund A. DiMarzio.	45
The cube of every connected graph is 1-hamiltonian. Gary Chartrand and S. F. Kapoor.	47
Minimum-length covering by intersecting intervals. W. A. Horn.	49
Publications of the National Bureau of Standards.	53

MIAMI UNIVERSITY LIBRARY

Library of Congress Catalog Card Number: 62–4414

For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 20402. Price 75 cents (single copy). Subscription price: $2.25 a year, $0.50 additional for foreign mailing.
UNITED STATES GOVERNMENT PRINTING OFFICE, WASHINGTON : 1969